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Bounds for the Solution of the Discrete Algebraic Lyapunov Equation*

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Key Words—Lyapunov; singular value decomposition.

Abstract—New bounds for solutions of the discrete algebraic Lyapunov equation $P = APA^{\mathrm{T}} + Q$ are derived. The new bounds are compared to existing ones and found to be of particular interest when A is non-normal. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

The discrete algebraic Lyapunov equation is

$$P = A^{\mathrm{T}}PA + Q \tag{1}$$

where P, A and Q are $n \times n$ matrices and Q is symmetric positive semi-definite. A unique, symmetric positive definite solution P exists when A is stable, i.e., when all the eigenvalues of A lie inside the unit circle. It is well known that solutions of equation (1) may be large compared to Q when eigenvalues of A have modulus near one (see e.g. Gajić and Qureshi, 1995, Chapter 3; Kwon, *et al.*, 1996). This behavior is apparent in the lower bound on the determinant of P from Komarrof (1992)

$$\det P \geq \det Q \prod_{i=1}^{n} (1 - |\lambda_i(A)|^2)^{-1}. \tag{2}$$

In this work we begin to extend this idea to the situation where the eigenvalues of A are well inside the unit circle but A is close to an unstable matrix; i.e., $A + E$ is unstable and $\|E\|$ is small. The eigenvalues of A can exhibit such sensitivity only when A is non-normal. A particular feature of non-normal stable matrices is that they may have large singular values.

Our motivation for seeking such bounds comes from the application of the Kalman filter to the problem of assimilating atmospheric data (e.g. Cohn and Parrish, 1991). Using a number of simplifying assumptions, the error covariance of the estimate of the state of the atmosphere satisfies equation (1) for appropriate choice of A and Q. Since the system comes from the discretization of a continuum problem, the dimension n is large, typically of the order 10^6, making any direct treatment of equation (1) impractical. Estimates of the solution of equation (1) can be used to investigate indirectly the dependence of P on A and Q and to develop approximate methods. In atmospheric dynamics, as in fluid dynamics, a dominant mechanism for instability is believed to be non-modal growth due to the underlying non-normal operators (Farrell and Ioannou, 1993; Trefethen *et al.*, 1993). When such non-modal growth is present, A is non-normal and has singular values greater than one. We present here a lower bound for the eigenvalues of P that is tighter than previous ones and shows the dependence of P on the singular values of A. We also present an upper bound for the eigenvalue of P that unlike the majority of upper bounds is applicable when the largest singular value of A is greater than one.

The notation $\sigma_i(A)$ denotes the ith singular value of A, $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$ and $\lambda_i(A)$ the ith eigenvalue, $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|$. The following inequalities are used in the proofs (Ostrowski, 1959):

Theorem 1. If X and Y are symmetric matrices then

$$\lambda_i(X) + \lambda_n(Y) \leq \lambda_i(X + Y) \leq \lambda_i(X) + \lambda_1(Y). \tag{3}$$

Theorem 2. If $X = X^{\mathrm{T}}$ and Y are square matrices then

$$\sigma_n^2(Y)\lambda_i(X) \leq \lambda_i(YXY^{\mathrm{T}}) \leq \sigma_1^2(Y)\lambda_i(X). \tag{4}$$

2. *Main results*

The main results are Theorem 3, a lower bound on the eigenvalues of P, Corollaries (4) and (5), lower bounds on the determinant and trace of P and Theorem (6), an upper bound on the eigenvalues of P.

Theorem 3. If P is the solution of equation (1) then

$$\lambda_i(P) \geq \lambda_n(Q)\left(1 + \frac{\sigma_i^2(A)}{1 - \sigma_n^2(A)}\right). \tag{5}$$

Some simple corollaries are:

Corollary 4. If P is a solution of equation (1) then

$$\mathrm{tr}\,\mathrm{P} \geq \lambda_n(\mathrm{Q})\left(n + \frac{\|A\|_{\mathrm{F}}^2}{1 - \sigma_n^2(A)}\right), \tag{6}$$

where $\|A\|_{\mathrm{F}} = \sum_{i=1}^{n} \sigma_i^2(A)$ is the usual Frobenius matrix norm.

Corollary 5. If P is a solution of equation (1) then

$$\det P \geq \lambda_n(Q)^n \prod_{i=1}^{n}\left(1 + \frac{\sigma_i^2(A)}{1 - \sigma_n^2(A)}\right). \tag{7}$$

Proof. From Theorem (1) applied to equation (1)

$$\lambda_i(P) \geq \lambda_n(Q) + \lambda_i(A^{\mathrm{T}}PA). \tag{8}$$

Let the Cholesky decomposition of P *be* $P = LL^{\mathrm{T}}$. Then

$$\lambda_i(A^{\mathrm{T}}PA) = \lambda_i(A^{\mathrm{T}}LL^{\mathrm{T}}A) = \lambda_i(L^{\mathrm{T}}AA^{\mathrm{T}}L). \tag{9}$$

Applying Theorem (2) gives

$$\lambda_i(AA^{\mathrm{T}})\sigma_n^2(L) \leq \lambda_i(L^{\mathrm{T}}AA^{\mathrm{T}}L) \leq \lambda_i(AA^{\mathrm{T}})\sigma_1^2(L). \tag{10}$$

However, if A is square then

$$\lambda_i(AA^{\mathrm{T}}) = \sigma_i^2(A) \tag{11}$$

and since P is positive definite symmetric

$$\lambda_i(P) = \sigma_i^2(L). \tag{12}$$

Thus

$$\sigma_i^2(A)\lambda_n(P) \leq \lambda_i(A^{\mathrm{T}}PA) \leq \sigma_i^2(A)\lambda_1(P). \tag{13}$$

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Using equation (13) gives,

$$\lambda_i(P) \geq \lambda_n(Q) + \sigma_i^2(A)\lambda_n(P). \quad (14)$$

In particular,

$$\lambda_n(P) \geq \frac{\lambda_n(Q)}{1 - \sigma_n^2(A)} \quad (15)$$

if $\sigma_n(A) < 1$ as is the case since $|\lambda_1(A)| < 1$. Substituting equation (15) into equation (14) gives equation (5).

Theorem 6. If P is a solution of equation (1) and $A = V^{-1}DV$ with $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ then

$$\lambda_i(P) \leq \lambda_i(Q) + \lambda_1(Q)\frac{\lambda_1^2(A)\kappa_2^2(V)}{1 - \lambda_1^2(A)} \quad (16)$$

where $\kappa_2(V) = \sigma_1(V)/\sigma_n(V)$.

Proof. The solution of equation (1) may be written

$$P = \sum_{n=0}^{\infty}(A^T)^n Q A^n. \quad (17)$$

Then,

$$\lambda_i(P) \leq \lambda_i(Q) + \lambda_1\left(\sum_{n=1}^{\infty}(A^T)^n Q A^n\right), \quad (18)$$

$$\lambda_i(P) \leq \lambda_i(Q) + \sum_{n=1}^{\infty}\lambda_1((A^T)^n Q A^n), \quad (19)$$

$$\lambda_i(P) \leq \lambda_i(Q) + \lambda_1(Q)\sum_{n=1}^{\infty}\sigma_1^2(A^n). \quad (20)$$

However,

$$A^n = V^{-1}D^nV, \quad (21)$$

$$\sigma_1(A^n) = \sigma_1(V^{-1}D^nV) \leq \sigma_1(V^{-1})\sigma_1(D^n)\sigma_1(V)$$
$$= \kappa_2(V)|\lambda_1(A)|^n, \quad (22)$$

where we use the fact that $\sigma_1(V^1) = \sigma_n(V)$. This implies

$$\lambda_i(P) \leq \lambda_i(Q) + \lambda_1(Q)\kappa_2^2(V)\sum_{n=1}^{\infty}|\lambda_1(A)|^{2n}. \quad (23)$$

or

$$\lambda_i(P) \leq \lambda_i(Q) + \lambda_1(Q)\kappa_2^2(V)\frac{|\lambda_1(A)|^2}{1 - |\lambda_1(A)|^2}. \quad (24)$$

3. *Remarks and comparisons to existing bounds*
Remark: If

$$\frac{\sigma_1^2(A) - \sigma_n^2(A)}{1 - \sigma_n^2(A)} > \frac{\lambda_i(Q)}{\lambda_n(Q)} - 1 \quad (25)$$

then Theorem 3 is a better lower bound than (Mori et al., 1982)

$$\lambda_i(P) \geq \lambda_i(Q) + \frac{\sigma_n^2(A)\lambda_n(Q)}{1 - \sigma_n^2(A)}. \quad (26)$$

For $Q = I$, inequality (25) is true. For general Q, Theorem 3 will give a better bound if $\sigma_1(A)$ is sufficiently large. Also, the nature of the bounds in Theorem 3 and in equation (26) is quite different; Theorem 3 emphasizes the dependence of P on A while equation (26) emphasizes its dependence on Q.

Remark 8. In general it is not simple to compare the bounds in equations (6) and (7) to known bounds (see e.g. Gajić and Qureshi, 1995 and references therein). We consider a specific physical example (Gajic and Qureshi, 1995, Example 3.8). Let A be

$$A = \begin{bmatrix} 0.4916 & 0 & 0 & 0 & 0 \\ 0 & 0.1353 & 0 & 0 & 0 \\ 0.2104 & 0.2283 & 0.2343 & 0.0319 & -0.0013 \\ -0.0086 & -0.0148 & 0.0316 & -0.4563 & -0.0164 \\ -0.3176 & -0.6624 & 1.8003 & 22.4148 & -0.4147 \end{bmatrix} \quad (27)$$

and $Q = I_5$. Then, the solution of equation (1) has $\text{tr}\,P = 1067.3097$ and $\det P = 11608.471$. The lower bound from Komarrof (1992)

$$\text{tr}\,P \geq n\left[\det Q\prod_{i=1}^{n}(1 - |\lambda_i(A)|^2)^{-1}\right]^{1/n} \quad (28)$$

gives $\text{tr}\,P \geq 7.409$, while Corollary (4) gives the bound $\text{tr}\,P \geq 512.185$. The bound in equation (2) gives $\det P \geq 7.144$, while Corollary (5) gives the bound $\det P \geq 737.122$.

Remark 9. If A is normal the bound of Theorem 6 reduces to the bound found in Mori *et al.* (1982)

$$\lambda_i(P) \leq \lambda_i(Q) + \lambda_1(Q)\frac{|\lambda_1(A)|^2}{1 - |\lambda_1(A)|^2}. \quad (29)$$

Remark 10. If A is non-normal the condition number of the eigenvectors of A, $\kappa_2(V)$ may be large as shown by the inequality (Loizou, 1969):

$$\kappa_2^2(V) \geq \left(1 + \frac{\Delta^2(A)}{\|A\|_F^2}\right), \quad (30)$$

where $\Delta(A)$ is defined as A's departure from normality (Golub and Van Loan, 1983).

Remark 11. Let A be a rank one matrix, i.e.

$$A = \sigma u v^T \quad (31)$$

with $\lambda = \sigma v^T u$ the eigenvalue of A and $0 < |\lambda| < 1$. Then the solution P is

$$P = Q + \frac{\sigma^2 v^T Q v}{1 - \lambda^2} u u^T. \quad (32)$$

If $Qu = \lambda_1(Q)u$ and $Qv = \lambda_1(Q)v$ then u is an eigenvector of P and

$$\lambda_1(P) = \lambda_1(Q)\left(1 + \frac{\sigma^2}{1 - \lambda^2}\right). \quad (33)$$

The sensitivity s of the non-zero eigenvalue of A is (Golub and Van Loan, 1983)

$$s = v^T u = \left(\frac{\lambda}{\sigma}\right); \quad (34)$$

roughly speaking a perturbation of size δ to A causes λ to change by δ/s. Thus,

$$\lambda_1(P) = \lambda_1(Q)\frac{1}{s^2}\frac{1 + (s^2 - 1)\lambda^2}{1 - \lambda^2}. \quad (35)$$

Thus, P is large compared to Q when either $\lambda \sim 1$ or when λ is sensitive ($s \sim 0$).

4. *Conclusions*

The aim of this work is to understand better the behavior of the solution of the discrete algebraic Lyapunov equation when A is non-normal and has large singular values. Here we derived a lower bound that shows the dependence of P on the singular values of A. We mention that a similar bound can be derived from matrix bounds in (Lee, 1996). We note that even for large matrices like the ones that arise in atmospheric data assimilation, iterative methods permit the computation of the leading singular values of A (Lacarra and Talagrand, 1988). Further work is needed to develop lower bounds that are appropriate for the case when A is non-normal and $|\lambda_1(A)|$ is near one.

As the survey in Kwon *et al.* (1996) shows, there are few upper bounds that can be used when the singular values of A are greater than one. The upper bound derived here depends on the conditioning of the eigenvectors of A which is in turn a measure of the sensitivity of of the eigenvalues of A. For the example in Remark 11, eigenvalue sensitivity plays an important role. However, eigenvector conditioning is often a crude indicator of eigenvalue sensitivity (Wilkinson, 1965). For this reason, the upper bound may fail to be tight and more investigation of the role of eigenvalue sensitivity is needed.

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