Some theoretical considerations on predictability of linear stochastic dynamics

Michael K. Tippett*
International Research Institute for climate prediction, Palisades, New York

Ping Chang
Department of Oceanography, Texas A& M University, Texas

February 1, 2002

*Corresponding author:IRI/LDEO 223 Monell, PO Box 1000 / 61 Route 9W, Palisades, NY 10964-8000. Phone: 845-680-4420. Fax: 845-680-4865. Email: tippett@iri.columbia.edu

Abstract

Predictability is a measure of prediction error relative to observed physical system variability. Here predictability of climate phenomena described by linear stochastic dynamics is considered. Prediction error obeys the same linear stochastic dynamics that govern the physical system when perfect initial conditions and perfect linear prediction dynamics are assumed. Prediction error growth depends on the choice of norm and on the interaction of the linear dynamics and stochastic forcing. Here predictability is measured using norm-independent multivariate generalizations of relative error. The predictability of linear stochastic dynamics is shown to be minimized when the stochastic forcing is uncorrelated in normal-mode space. Minimum predictability depends only on the eigenvalues of the dynamics and is a lower bound for the predictability of systems where the stochastic forcing has arbitrary spatial structure. Issues related to upper bounds for predictability are explored in a simple theoretical example.

1. Introduction

The original work of Hasselmann (1976) on stochastic climate theory pioneered the use of linear stochastic dynamics for modeling and predicting various modes of climate variability. Since then, evidence has been presented that many climate phenomena are described by linear stochastic dynamics, at least as a first-order approximation (Penland and Matrosova, 1994; Whitaker and Sardeshmukh, 1998). Consequently, linear stochastic models are routinely used in operational climate forecasts (Penland and Matrosova, 1998; Winkler et al., 2001). Therefore, a general understanding of the predictability of phenomena described by linear stochastic dynamics is of theoretical interest and practical value.

A system is predictable on those time-scales where prediction errors do not exceed some predetermined fraction of climate variability (Lorenz, 1969). The ratio of prediction error variance to observed climatological variance is a measure of predictability in univariate systems with Gaussian distributions. This notion of predictability based on relative error can be extended to multivariate systems using the *predictive information matrix* (Schneider and Griffies, 1999; Kleeman, 2001). The eigenvalues of the predictive information matrix can be used to construct multivariate generalizations of relative prediction error. Importantly, these predictability measures are norm-independent, unlike the commonly used predictive error variance.

Prediction error evolution has the form of linear stochastic dynamics when the physical system is described by linear stochastic dynamics and the prediction system has linear deterministic dynamics. Prediction error growth depends only on the response of the prediction system dynamics to stochastic forcing when the stochastic forcing is state-independent and the initial conditions are perfect. Previous studies of similar error dynamics have found the stochastic forcing structures that maximize prediction error growth (stochastic optimals) and have considered the prediction error response to homogeneous, spatially uncorrelated stochastic forcing (Farrell and Ioannou, 1996;

Kleeman and Moore, 1997). More recently, Chang et al. (2002) identified the forcing pattern that maximizes the predictability of a linear stochastic system under an error variance based prediction measure. A limitation of such results is that measures of prediction error are not invariant with respect to linear transformations of the state-variable and are norm-dependent.

Here we examine predictability of linear stochastic dynamics general using norm-independent predictability measures and consider systems where the stochastic forcing has general spatial structure. These two considerations are related since setting the stochastic forcing to be spatially uncorrelated and homogeneous requires fixing a norm. We determine a lower-bound for predictability independent of the stochastic forcing and show that this minimum predictability is achieved when the stochastic forcing is uncorrelated in normal-mode space. Stochastic forcing with arbitrary spatial structure increases predictability.

We begin our discussion in Section 2 with an introduction to a linear stochastic system and its associated prediction error; in Section 3 we analyze the predictability of the linear stochastic system under a set of general predictability measures; in Section 4 we use a theoretical example to explore open issues related to upper bounds for predictability; in Section 5 we summarize our findings and discuss their implications.

2. Linear stochastic dynamics

We assume the observed phenomenon of interest is represented by a real n-dimensional state-vector $\mathbf{w}^{\text{obs}}$ whose evolution is governed by linear stochastic dynamics. That is to say, we assume the observed state $\mathbf{w}^{\text{obs}}$ satisfies

$$\frac{d\mathbf{w}^{\text{obs}}}{dt} = \mathbf{A}\mathbf{w}^{\text{obs}} + \mathbf{F}\xi\,, \quad \mathbf{w}^{\text{obs}}(t=0) = \mathbf{w}_0^{\text{obs}}\,, \tag{1}$$

where the dynamics matrix $\mathbf{A}$ and the forcing matrix $\mathbf{F}$ are constant, real $n \times n$ matrices; ξ is n-dimensional, spatially uncorrelated Gaussian white-noise, $\langle \xi(t_1)\xi(t_2)^T \rangle = \delta(t_1 - t_2)\mathbf{I}$ where the

notation $\langle\cdot\rangle$ denotes ensemble average and $()^T$ denotes matrix transpose. A general deterministic prediction system has the form

$$\frac{d\mathbf{w}^{\text{pred}}}{dt} = \mathbf{A}^{\text{pred}}\mathbf{w}^{\text{pred}}\,, \quad \mathbf{w}^{\text{pred}}(t=0) = \mathbf{w}_0^{\text{pred}}\,. \tag{2}$$

Differences between the observed state $\mathbf{w}^{\text{obs}}$ and the predicted state $\mathbf{w}^{\text{pred}}$ are due to (*i*) differences between the observed initial condition $\mathbf{w}_0^{\text{obs}}$ and the prediction initial condition $\mathbf{w}_0^{\text{pred}}$, (*ii*) deficiencies in the prediction dynamics $\mathbf{A}^{\text{pred}}$ and, (*iii*) the presence of stochastic processes in the observations. This dependence is explicit in the equation for the evolution of the prediction error $\mathbf{w} \equiv \mathbf{w}^{\text{obs}} - \mathbf{w}^{\text{pred}}$

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}^{\text{pred}}\mathbf{w} + (\mathbf{A} - \mathbf{A}^{\text{pred}})\mathbf{w}^{\text{obs}} + \mathbf{F}\xi\,, \quad \mathbf{w}(t=0) = \mathbf{w}_0^{\text{obs}} - \mathbf{w}_0^{\text{pred}}\,. \tag{3}$$

The forcing in the prediction error dynamics consists of two components. The first component $(\mathbf{A} - \mathbf{A}^{\text{pred}})\mathbf{w}^{\text{obs}}$ represents the error due to imperfect deterministic dynamics of the prediction model and depends on the observed state; the second component $\mathbf{F}\xi$ represents the unpredictable stochastic processes in the observations and is state-independent. We take the forcing to be state-independent and assume that the prediction system has perfect dynamics $\mathbf{A}^{\text{pred}} = \mathbf{A}$. Additionally we assume perfect initial conditions $\mathbf{w}_0^{\text{pred}} = \mathbf{w}_0^{\text{obs}}$. Therefore, the source of the prediction error is entirely due to the stochastic processes. Chang et al. (2002) refer to this situation as the *perfect initial condition* scenario and give a more detailed discussion. In this scenario, the prediction error $\mathbf{w}$ evolves according to

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}\mathbf{w} + \mathbf{F}\xi\,, \quad \mathbf{w}(t=0) = 0\,. \tag{4}$$

We take the dynamics matrix $\mathbf{A}$ to be stable, i.e., all its eigenvalues $\lambda_k(\mathbf{A})$ have negative real part. We use the convention that the eigenvalues of $\mathbf{A}$ are ordered least damped to most damped so that $0 > \operatorname{Re}\lambda_1(\mathbf{A}) \geq \operatorname{Re}\lambda_2(\mathbf{A}) \cdots \geq \operatorname{Re}\lambda_n(\mathbf{A})$.

The prediction error covariance at lead-time τ is defined as $\mathbf{C}_\tau \equiv \langle \mathbf{w}(\tau)\mathbf{w}(\tau)^T \rangle$ and is well-described by its eigenvectors and eigenvalues. The eigenvectors or EOFs of the prediction error

covariance are orthogonal and order state-space according to the amount of variance they explain; $\lambda_k(\mathbf{C}_\tau)$ is the variance explained by the k-th eigenvector of $\mathbf{C}_\tau$; $\lambda_1(\mathbf{C}_\tau) \geq \lambda_2(\mathbf{C}_\tau) \geq \cdots \geq \lambda_n(\mathbf{C}_\tau) \geq 0$. Since orthogonality depends on the choice of norm, or equivalently on the choice of state-variable, the eigenvalue decomposition of the prediction error covariance is not invariant under linear transformations of the state-variable. If we define a new state-variable $\hat{\mathbf{w}} = \mathbf{L}\mathbf{w}$ where $\mathbf{L}$ is a linear transformation, the transformed prediction error covariance $\hat{\mathbf{C}}_\tau \equiv \langle \hat{\mathbf{w}}(\tau)\hat{\mathbf{w}}(\tau)^T \rangle$, is given by $\hat{\mathbf{C}}_\tau = \mathbf{L}\mathbf{C}_\tau\mathbf{L}^T$. The prediction error covariance matrices $\mathbf{C}_\tau$ and $\hat{\mathbf{C}}_\tau$ have the same eigenvalues only when $\mathbf{L}$ is an orthogonal transformation, in which case the transformation $\mathbf{L}$ relates eigenvectors of $\mathbf{C}_\tau$ to eigenvectors of $\hat{\mathbf{C}}_\tau$. Measures of prediction error growth that depend on the eigenvalues of $\mathbf{C}_\tau$, for instance, the total variance $\mathrm{tr}\,\mathbf{C}_\tau$, depend on the choice of state-variable.

For the prediction error dynamics in (4), the prediction error covariance is

$$\mathbf{C}_\tau = \int_0^\tau e^{t\mathbf{A}}\mathbf{F}\mathbf{F}^T e^{t\mathbf{A}^T}\, dt\,. \tag{5}$$

Suppose the dynamics matrix $\mathbf{A}$ is diagonalizable with eigendecomposition $\mathbf{A} = \mathbf{Z}\Lambda\mathbf{Z}^{-1}$; the matrix Λ of eigenvalues is an $n \times n$ diagonal matrix whose k-th diagonal entry is $\lambda_k(\mathbf{A})$; the k-th column of the $n \times n$ matrix $\mathbf{Z}$ is the eigenvector $\mathbf{z}_k$ and satisfies $\mathbf{A}\mathbf{z}_k = \lambda_k(\mathbf{A})\mathbf{z}_k$. The matrix $\mathbf{Y}$ of adjoint eigenvectors of $\mathbf{A}$ is defined by $\mathbf{Y} \equiv (\mathbf{Z}^{-1})^\dagger$ where the notation $()^\dagger$ denotes conjugate transpose. We assume without loss of generality that the columns $\mathbf{y}_k$ of $\mathbf{Y}$ are unit vectors with $\mathbf{y}_k^\dagger\mathbf{y}_k = 1$. Now the prediction error covariance $\mathbf{C}_\tau$ can be expressed in the basis of the eigenvectors of the dynamics as $\mathbf{C}_\tau = \mathbf{Z}\tilde{\mathbf{C}}_\tau\mathbf{Z}^\dagger$. Using the relation $\mathbf{Y}^\dagger\mathbf{Z} = \mathbf{Z}^\dagger\mathbf{Y} = \mathbf{I}$, the matrix $\tilde{\mathbf{C}}_\tau$ is determined by

$$\begin{aligned} \tilde{\mathbf{C}}_\tau = \mathbf{Y}^\dagger\mathbf{C}_\tau\mathbf{Y} &= \int_0^\tau e^{\Lambda t}\mathbf{Y}^\dagger\mathbf{F}\mathbf{F}^T\mathbf{Y}e^{\Lambda^\dagger t}\, dt \\ &= \mathbf{Y}^\dagger\mathbf{F}\mathbf{F}^T\mathbf{Y} \circ \mathbf{E}_\tau\,, \end{aligned} \tag{6}$$

where the notation $\circ$ denotes Hadamard product[1], and the entries of the positive semidefinite matrix

[1]The Hadamard product of two matrices $\mathbf{X}$ and $\mathbf{Y}$ with entries $\mathbf{X}_{kl}$ and $\mathbf{Y}_{kl}$, respectively, is the matrix whose entries are $\mathbf{X}_{kl}\mathbf{Y}_{kl}$.

$\mathbf{E}_\tau$ are

$$[\mathbf{E}_\tau]_{kl} = \int_0^\tau e^{(\lambda_k(\mathbf{A})+\overline{\lambda}_l(\mathbf{A}))t}\, dt = \frac{e^{(\lambda_k(\mathbf{A})+\overline{\lambda}_l(\mathbf{A}))\tau} - 1}{\lambda_k(\mathbf{A}) + \overline{\lambda}_l(\mathbf{A})}\,. \tag{7}$$

The matrix $\mathbf{E}_\tau$ depends only on the eigenvalues of the dynamics and the lead-time τ. The projection $\mathbf{P}$ of the forcing matrix $\mathbf{F}$ onto the adjoint eigenvectors $\mathbf{Y}$ is defined by $\mathbf{P} \equiv \mathbf{F}^T\mathbf{Y}$. The representation $\mathbf{C}_\tau = \mathbf{Z}\left(\mathbf{P}^\dagger\mathbf{P} \circ \mathbf{E}_\tau\right)\mathbf{Z}^\dagger$ shows the dependence of the prediction error covariance on the eigendecomposition of the dynamics, as well as on the projection $\mathbf{P}$ of the forcing matrix onto the adjoint eigenvectors.

The dependence of the prediction error covariance on the dynamics is particularly clear when the stochastic forcing is uncorrelated in normal-mode space, that is, when $\mathbf{F}\mathbf{F}^T = \mathbf{Z}\mathbf{D}\mathbf{Z}^\dagger$ and $\mathbf{D}$ is diagonal. This characterization of the forcing matrix is invariant with respect to linear transformations of the state-variable. When the stochastic forcing is uncorrelated in normal-model space, the adjoint eigenvectors of the dynamics diagonalize the prediction error covariance at all lead times, i.e., normal modes are uncorrelated, and $\mathbf{C}_\tau = \mathbf{Z}(\mathbf{D}\circ\mathbf{E}_\tau)\mathbf{Z}^\dagger$. Note that that $\mathbf{D}\circ\mathbf{E}_\tau$ is a diagonal matrix whose k-th diagonal entry is $\mathbf{D}_{kk}[\mathbf{E}_\tau]_{kk}$. Although this representation separates temporal and spatial structures of the prediction error covariance, it is not the eigenvalue decomposition since the matrix of eigenvectors $\mathbf{Z}$ is generally not orthogonal. Lower bounds for the largest eigenvalue and the total prediction error variance in the case of normal-mode uncorrelated stochastic forcing are

$$\lambda_1(\mathbf{C}_\tau) \geq \max_{1\leq k\leq n} \mathbf{y}_k^\dagger\mathbf{C}_\tau\mathbf{y}_k = \max_{1\leq k\leq n} \frac{e^{2\,\mathrm{Re}\,\lambda_k(\mathbf{A})\tau} - 1}{2\,\mathrm{Re}\,\lambda_k(\mathbf{A})}\mathbf{D}_{kk}\,, \tag{8}$$

$$\begin{aligned} \mathrm{tr}\,\mathbf{C}_\tau &= \sum_{k=1}^n \frac{e^{2\,\mathrm{Re}\,\lambda_k(\mathbf{A})\tau} - 1}{2\,\mathrm{Re}\,\lambda_k(\mathbf{A})}\mathbf{D}_{kk}\mathbf{z}_k^\dagger\mathbf{z}_k \\ &\geq \sum_{k=1}^n \frac{e^{2\,\mathrm{Re}\,\lambda_k(\mathbf{A})\tau} - 1}{2\,\mathrm{Re}\,\lambda_k(\mathbf{A})}\mathbf{D}_{kk}\,. \end{aligned} \tag{9}$$

The second inequality follows from $\mathbf{z}_k^\dagger\mathbf{z}_k = \sec^2\theta_k \geq 1$ where θ_k is the angle between the eigenvector $\mathbf{z}_k$ and the adjoint eigenvector $\mathbf{y}_k$.[2] Consequently the prediction error variance is large when

[2]Recall that $\sec\theta_k = \|\mathbf{z}_k\|\|\mathbf{y}_k\|/\mathbf{y}_k^\dagger\mathbf{z}_k$ and $\|\mathbf{y}_k\| = \mathbf{y}_k^\dagger\mathbf{z}_k = 1$.

$\operatorname{Re}\lambda_k(\mathsf{A})$ is small and when $\mathbf{z}_k$ and $\mathbf{y}_k$ are nearly orthogonal.

The inequalities in (8) and (9) are equalities when the dynamics matrix A is normal and the stochastic forcing is uncorrelated in normal-mode space. Spatially uncorrelated stochastic forcing is also uncorrelated or white in normal-mode space for normal dynamics. Moreover, in this case, the eigenvalues of the prediction error covariance are given by

$$\lambda_k(\mathsf{C}_\tau) = \lambda_k(\operatorname{diag}\mathsf{E}_\tau) = \frac{e^{2\operatorname{Re}\lambda_k(\mathsf{A})\tau} - 1}{2\operatorname{Re}\lambda_k(\mathsf{A})}\mathsf{D}_{kk}\,, \tag{10}$$

recalling that $\operatorname{Re}\lambda_k(\mathsf{A}) < 0$; $\operatorname{diag}\mathsf{E}_\tau$ is the diagonal matrix whose entries are given by the diagonal elements of E_τ. Consequently, useful measures of the size of the prediction error covariance C_τ, such as $\operatorname{tr}\mathsf{C}_\tau$ and $\det\mathsf{C}_\tau$, are entirely determined by the eigenvalues of the dynamics matrix A and the forcing coefficients D_{kk} when the dynamics is normal. Additionally, the eigenvectors of the covariance matrix C_τ are closely related to the eigenvectors of the dynamics. When $\mathbf{z}_k$ is real (or equivalently $\lambda_k(\mathsf{A})$ is real), $\mathbf{z}_k$ is the k-th eigenvector of C_τ; when $\lambda_k(\mathsf{A})$ is not real, C_τ has a repeated eigenvalue and two corresponding, nonunique eigenvectors $\operatorname{Re}\mathbf{z}_k$ and $\operatorname{Im}\mathbf{z}_k$.

Since details of the stochastic forcing are seldom known, it is useful to know which forcing structures most efficiently excite error growth. Generally, the most efficient stochastic forcing is not uncorrelated in normal-mode space and stochastic forcing with arbitrary spatial structure must be considered. For instance, the ratio $\lambda_1(\mathsf{C}_\tau)/\lambda_1(\mathsf{F}\mathsf{F}^T)$ is largest when $\mathsf{F}\mathsf{F}^T = \mathsf{I}$ (Tippett and Cohn, 2001). The ratio of prediction error variance to stochastic forcing variance $\operatorname{tr}\mathsf{C}_\tau/\operatorname{tr}\mathsf{F}\mathsf{F}^T$ is largest when F has one nonzero column that is the leading eigenvector of the matrix B_τ defined by

$$\mathsf{B}_\tau \equiv \int_0^\tau e^{t\mathsf{A}^T} e^{t\mathsf{A}}\, dt\,, \tag{11}$$

that is, when F is the leading stochastic optimal. These two choices of forcing are uncorrelated in normal-mode space only when A is normal. Therefore, maximal prediction error growth in nonnormal systems is due to forcing with general spatial structure. Unfortunately, prediction error due to arbitrary forcing is less simply described and analyzed.

The remarks here show that prediction error growth depends strongly on details of the forcing structure, particularly the projection of the forcing onto adjoint eigenmodes of the dynamics. Therefore prediction error growth due to normal-mode uncorrelated forcing is simply described. However, normal-mode uncorrelated forcing does not in general include the most (or least) efficient forcing. Additionally, prediction error growth measures depend on the choice of norm.

3. Predictability

A prediction is useful on time-scales where the prediction error is, in some sense, less than the climatological variability. Therefore, the utility of a prediction at lead-time τ depends on both the prediction error covariance C_τ and the climatological error covariance C_∞. This notion of predictability is the basis for predictability measures defined using the *predictive information matrix* $\mathsf{G}_\tau \equiv \mathsf{C}_\tau \mathsf{C}_\infty^{-1}$ (Schneider and Griffies, 1999).[3] The eigenvalues of the predictive information matrix give a multivariate generalization of the univariate relative error variance s_τ^2/s_∞^2 where s_τ^2 is the prediction error variance at lead-time τ and s_∞^2 is the climatological variance. The eigenvalues of the predictive information matrix can be used to construct multivariate analogs of the univariate predictability measure, $1 - s_\tau^2/s_\infty^2$.

The eigenvalue decomposition of the predictive information G_τ decomposes phase-space into uncorrelated patterns ordered by their relative prediction error. When λ is an eigenvalue of G_τ, its adjoint eigenvector $\mathbf{q}$ satisfies $\mathbf{q}^T\mathsf{G}_\tau = \lambda\mathbf{q}^T$. The eigenvalues of G_τ are between zero and unity since

$$\lambda = \frac{\mathbf{q}^T\mathsf{C}_\tau\mathbf{q}}{\mathbf{q}^T\mathsf{C}_\infty\mathbf{q}}\,, \tag{12}$$

and $\mathbf{q}^T\mathsf{C}_\infty\mathbf{q} \geq \mathbf{q}^T\mathsf{C}_\tau\mathbf{q} > 0$. The eigenvalues of G_τ behave like relative error, initially zero because of the perfect initial condition assumption, and increasing with lead-time until they reach

[3] The climatological covariance matrix is invertible, i.e., there are no perfectly predictable components, if the pair (A, F) is controllable; a sufficient condition for controllability is that forcing covariance $\mathsf{F}\mathsf{F}^T$ be invertible.

unity in the limit of large lead-time. The quantity $(\mathbf{q}^T\mathbf{C}_\tau\mathbf{q})/(\mathbf{q}^T\mathbf{C}_\infty\mathbf{q})$ is the relative error of the linear combination $(\mathbf{q}^T\mathbf{w})$ of state-variable elements. The weight $\mathbf{q}_n$ that minimizes the relative error in (12) is the n-th adjoint eigenvector of the predictive information matrix and defines the first *predictable component* $(\mathbf{q}_n^T\mathbf{w})$ and its relative error variance $\lambda_n(\mathbf{G}_\tau)$ (Schneider and Griffies, 1999). The eigenvector $\mathbf{p}_n$ corresponding to the adjoint eigenvector $\mathbf{q}_n$ is the first *predictable pattern* (Schneider and Griffies, 1999). The second predictable component is determined by the next smallest eigenvalue of $\mathbf{G}_\tau$ and is temporally uncorrelated with the first predictable component.

The eigenvalues of the predictive information matrix are invariant with respect to linear transformations of the state-variable. If a new state-variable $\hat{\mathbf{w}} \equiv \mathbf{L}\mathbf{w}$ and its prediction error covariance $\hat{\mathbf{C}}_\tau$ is defined, the new predictive information matrix $\hat{\mathbf{G}}_\tau$ is related to $\mathbf{G}_\tau$ by a similarity transformation

$$\hat{\mathbf{G}}_\tau = \hat{\mathbf{C}}_\tau\hat{\mathbf{C}}_\infty^{-1} = \mathbf{L}\mathbf{C}_\tau\mathbf{C}_\infty^{-1}\mathbf{L}^{-1} = \mathbf{L}\mathbf{G}_\tau\mathbf{L}^{-1}\,, \tag{13}$$

so that $\mathbf{G}_\tau$ and $\hat{\mathbf{G}}_\tau$ have the same eigenvalues. Therefore, predictability measures defined by eigenvalues of the predictive information matrix are invariant with respect to linear transformations of the state-variable and are norm-independent. The eigenvectors of the predictive information matrix transform in the same manner as the state-variable; if $\mathbf{p}$ is an eigenvector of $\mathbf{G}_\tau$ then $\mathbf{L}\mathbf{p}$ is an eigenvector of $\hat{\mathbf{G}}_\tau$.

The predictable patterns and their relative error variances are simply related to the dynamics when the forcing is uncorrelated in normal-mode space, i.e., when $\mathbf{F}\mathbf{F}^T = \mathbf{Z}\mathbf{D}\mathbf{Z}^\dagger$ and $\mathbf{D}$ is diagonal. We require that $\mathbf{D}$ be invertible to insure that the climatological covariance $\mathbf{C}_\infty$ is also invertible. In this case, the predictive information matrix $\mathbf{G}_\tau$ has the simple form

$$\begin{aligned}\mathbf{G}_\tau &= \mathbf{Z}(\mathbf{D}\circ\mathbf{E}_\tau)\mathbf{Z}^\dagger\left(\mathbf{Z}^\dagger\right)^{-1}(\mathbf{D}\circ\mathbf{E}_\infty)^{-1}\mathbf{Z}^{-1} \\ &= \mathbf{Z}\,\mathrm{diag}(\mathbf{E}_\tau)\,\mathrm{diag}(\mathbf{E}_\infty)^{-1}\mathbf{Z}^{-1}\,,\end{aligned} \tag{14}$$

and is remarkably independent of the forcing coefficient $\mathbf{D}$. Since (14) is the eigendecomposition

of G_τ, the eigenvalues of G_τ are

$$\lambda_k(\mathsf{G}_\tau) = 1 - e^{2\,\mathrm{Re}\,\lambda_{n-k+1}(\mathsf{A})\tau}\,, \tag{15}$$

and depend only on the real part of the eigenvalues of the dynamics. The eigenvectors of G_τ are the eigenvectors of the dynamics. Therefore, the leading eigenvector $\mathbf{z}_1$ of the dynamics is the first predictable pattern at all lead-times of a system with normal-mode uncorrelated forcing; the relative error of the associated first predictable component is $1 - e^{2\,\mathrm{Re}\,\lambda_1(\mathsf{A})}$. The predictability of the first predictable component decreases exponentially with decay-rate determined by the least damped eigenmode of the dynamics.

For general forcing, it is useful to write the prediction error covariance C_τ as

$$\begin{aligned}\mathsf{C}_\tau &= \int_0^\infty e^{t\mathsf{A}}\mathsf{F}\mathsf{F}^T e^{t\mathsf{A}^T}\,dt - \int_\tau^\infty e^{t\mathsf{A}^\dagger}\mathsf{F}\mathsf{F}^T e^{t\mathsf{A}^T}\,dt\,,\\ &= \mathsf{C}_\infty - e^{\tau\mathsf{A}}\mathsf{C}_\infty e^{\tau\mathsf{A}^T}\,,\end{aligned} \tag{16}$$

so that

$$\mathsf{G}_\tau = \mathsf{I} - e^{\mathsf{A}\tau}\mathsf{C}_\infty e^{\mathsf{A}^T\tau}\mathsf{C}_\infty^{-1}\,. \tag{17}$$

The eigenvalues of G_τ are then

$$\begin{aligned}\lambda_k(\mathsf{G}_\tau) &= \lambda_k(\mathsf{I} - \mathsf{W}_\tau\mathsf{W}_\tau^T)\\ &= 1 - \sigma_{n-k+1}^2(\mathsf{W}_\tau)\,,\end{aligned} \tag{18}$$

where we define $\mathsf{W}_\tau \equiv e^{\hat{\mathsf{A}}\tau}$ and $\hat{\mathsf{A}} \equiv \mathsf{C}_\infty^{-1/2}\mathsf{A}\mathsf{C}_\infty^{1/2}$; the notation $\sigma_k(\mathsf{X})$ denotes the k-th singular value of a matrix X ordered so that $\sigma_1(\mathsf{X}) \geq \sigma_2(\mathsf{X}) \geq \ldots \sigma_n(\mathsf{X}) \geq 0$. The matrix $\hat{\mathsf{A}}$ is the dynamics matrix of the whitened state-variable $\hat{\mathbf{w}} \equiv \mathsf{C}_\infty^{-1/2}\mathbf{w}$. The climatological covariance $\hat{\mathsf{C}}_\infty$ is the identity matrix in the whitened state-variable; W_τ is the state-propagator of the whitened state-variable. The eigendecomposition of G_τ is determined by the singular value decomposition of W_τ. Another way of arriving at (18) is to recall that the eigenvalues of G_τ are invariant under linear transformations of the state-variable and note that in the whitened state-variable

$$\hat{\mathsf{G}}_\tau = \hat{\mathsf{C}}_\tau = \mathsf{I} - \mathsf{W}_\tau\mathsf{W}_\tau^T\,. \tag{19}$$

Predictability analysis is equivalent to principle component analysis of the whitened state-vector $\hat{\mathbf{w}}$ (Schneider and Griffies, 1999).

The whitened dynamics $\hat{\mathbf{A}}$ is normal when the forcing is uncorrelated in normal-mode space. In this case, the whitened-state propagator $\mathbf{W}_\tau$ is also normal, $\sigma_k^2(\mathbf{W}_\tau) = |\lambda_k(\mathbf{W}_\tau)|^2 = e^{2\,\mathrm{Re}\,\lambda_k(\mathbf{A})\tau}$, and (15) is a consequence. When the forcing is correlated in normal-mode space, $\mathbf{W}_\tau$ is nonnormal and its singular values are not determined by its eigenvalues. However, the eigenvalues and singular values of any matrix must satisfy certain inequalities. For instance, the largest singular value must be larger than the modulus of the largest eigenvalue. That is to say, $\sigma_1^2(\mathbf{W}_\tau) \geq |\lambda_1(\mathbf{W}_\tau)|^2$, which combined with (18) gives the upper bound

$$\lambda_n(\mathbf{G}_\tau) \leq 1 - e^{2\,\mathrm{Re}\,\lambda_1(\mathbf{A})\tau}\,, \tag{20}$$

for the smallest eigenvalue of the predictive information matrix. This upper bound means that the relative error of the first predictable component is always less than the relative error of the first predictable component when the stochastic forcing is uncorrelated in normal-mode space. In other words, the relative error of the first predictable component is maximized when the forcing is uncorrelated in normal-mode space. When the stochastic forcing is correlated in normal-mode space, normal modes are correlated and linear combinations of normal modes can be constructed whose relative error is less than the relative error of any single normal mode. Therefore, normal-mode uncorrelated forcing minimizes predictability as measured by the relative error of the first predictable component.

The same conclusion that normal-mode uncorrelated stochastic forcing minimizes predictability holds for general predictability measures defined abstractly by

$$\text{predictability} \equiv \sum_{k=1}^{n} h(1 - \lambda_k(\mathbf{G}_\tau)) = \sum_{k=1}^{n} h(\sigma_k^2(\mathbf{W}_\tau))\,, \tag{21}$$

where the function h has the property that $h(e^x)$ is a convex and increasing function of x. Theorem

1 of the Appendix and (18) give that

$$\text{predictability} = \sum_{k=1}^{n} h(\sigma_k^2(\mathbf{W}_\tau)) \geq \sum_{k=1}^{n} h(|\lambda_k^2(\mathbf{W}_\tau)|) = \sum_{k=1}^{n} h\left(e^{2\,\mathrm{Re}\,\lambda_k(\mathbf{A})}\right) . \tag{22}$$

The right-hand side of (22) is the predictability of the system with normal-mode uncorrelated forcing and depends only the the real part of the eigenvalues of the dynamics. General predictability measures are minimized when the stochastic forcing is uncorrelated in normal-mode space.

Examples of predictability measures that can be written in the form of (21) are the quantity

$$1 - \mathrm{tr}\,\mathbf{G}_\tau/n = \frac{1}{n}\sum_{k=1}^{n} \sigma_k^2\left(\mathbf{W}_\tau\right) , \tag{23}$$

the *predictive information* R_τ, defined by (Schneider and Griffies, 1999)

$$R_\tau \equiv -\frac{n}{2}\log\det\mathbf{G}_\tau = -\frac{n}{2}\sum_{k=1}^{n}\log\left(1-\sigma_k^2(\mathbf{W}_\tau)\right) , \tag{24}$$

and *relative entropy* r_τ defined by (Kleeman, 2001)

$$r_\tau \equiv \frac{1}{2}\left[-\log\left(\det\mathbf{G}_\tau\right) + \mathrm{tr}\left(\mathbf{G}_\tau\right) - n\right] = -\frac{1}{2}\sum_{k=1}^{n}\log\left(1-\sigma_k^2(\mathbf{W}_\tau)\right) + |\lambda_k(\mathbf{W}_\tau)|^2 . \tag{25}$$

Normal-mode uncorrelated forcing minimizes all these measures of predictability and gives lower bounds for predictability that depend only on the eigenvalues of the dynamics (see Lemmas 1 and 2 of the Appendix). The same statement is true when the sums in (23) – (24) are truncated, and predictability is measured using the leading eigenvalues of the predictive information matrix.

These results have an interpretation in the framework where the forcing is fixed as homogeneous and spatially uncorrelated, i.e., when $\mathbf{F}\mathbf{F}^T = \mathbf{I}$. In this case, the stochastic forcing is uncorrelated in normal-mode space when $\mathbf{Y}^\dagger\mathbf{Y}$ is diagonal. When $\mathbf{Y}^\dagger\mathbf{Y}$ is diagonal, $\mathbf{Y}^\dagger\mathbf{Y} = \mathbf{I}$ and the dynamics are normal since the columns of $\mathbf{Y}$ are unit vectors. Therefore, for homogeneous and spatially uncorrelated forcing, predictability is minimized when the dynamics is normal. Non-normal dynamics is more predictable than normal dynamics with the same eigenvalues when the stochastic forcing is homogeneous and spatially uncorrelated.

4. Upper bounds for predictability

Our analysis shows that the predictability of a system described by linear stochastic dynamics is minimized when the stochastic forcing is uncorrelated in normal-mode space. However, some questions remain unanswered. For instance, what stochastic forcing produces maximum predictability and when is this maximum predictability strictly larger than that given by normal-mode uncorrelated forcing? Such questions are difficult to answer generally because predictability is a nonlinear function of the stochastic forcing. We explore these issues in a simple 2-D linear stochastic system. The relevance of this type of simple model to real climate systems is discussed in Chang et al. (2002) where a 2-D stochastically driven damped inertial oscillator was used as a prototype coupled system to illustrate the importance of nonnormal growth in enhancing the predictability of climate systems.

Consider the 2×2 dynamics matrix given by

$$\mathbf{A} = \begin{bmatrix} \lambda_1 + i\phi & 0 \\ 0 & \lambda_2 - i\phi \end{bmatrix} , \tag{26}$$

where λ_1 and λ_2 are negative real constants; ϕ is a real constant and $i = \sqrt{-1}$. We assume that the two eigenvalues of $\mathbf{A}$ are either both real or complex conjugates. There is no loss of generality in taking the dynamics to be diagonal since our predictability measures are invariant linear transformation of the state variable. Normal-mode uncorrelated stochastic forcing for diagonal dynamics corresponds to $\mathbf{F}\mathbf{F}^T$ being diagonal. In this case, the predictive information matrix $\mathbf{G}_\tau$ is diagonal and given by

$$\mathbf{G}_\tau = \begin{bmatrix} 1 - e^{2\lambda_1 \tau} & 0 \\ 0 & 1 - e^{2\lambda_2 \tau} \end{bmatrix} , \tag{27}$$

and can be used to compute the minimum predictability of the system. When the eigenvalues of the dynamics are identical, $\lambda \equiv \lambda_1 = \lambda_2$ and $\phi = 0$, the dynamics is essentially scalar and

$$\mathbf{C}_\tau = \frac{1 - e^{2\lambda\tau}}{2\lambda} \mathbf{F}\mathbf{F}^T , \tag{28}$$

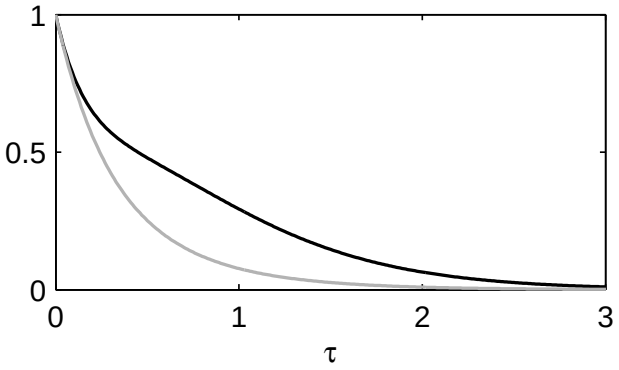

Figure 1. The predictability measure $1 - \mathrm{tr}\, \mathsf{G}_\tau/2$ for normal-mode uncorrelated forcing (gray line) and rank-1 forcing (black line) with $\lambda_1 = -1$ and $\lambda_2 = -2$.

for general stochastic forcing. In this case, the climatological covariance $\mathsf{C}_\infty = \mathsf{F}\mathsf{F}^T/(2\lambda)$ is invertible when $\mathsf{F}\mathsf{F}^T$ is, and the predictive information matrix is independent of the forcing and given by (27) with $\lambda_1 = \lambda_2$. Therefore no forcing structure increases predictability when the eigenvalues of the dynamics are identical, a result that is true in general.

Suppose that the forcing matrix is rank-1 so that the stochastic forcing covariance can be written $\mathsf{F}\mathsf{F}^T = \mathbf{f}\mathbf{f}^T$ where $\mathbf{f}$ is a vector with nonzero elements. A general result that follows directly from (6) is that

$$\mathsf{C}_\tau = \mathbf{f}\mathbf{f}^T \circ \mathsf{E}_\tau = \mathrm{Diag}(\mathbf{f})\, \mathsf{E}_\tau\, \mathrm{Diag}(\mathbf{f})\,, \tag{29}$$

when the forcing matrix F is rank-1 and the dynamics are diagonal; the notation $\mathrm{Diag}(\mathbf{f})$ denotes the diagonal matrix whose diagonal entries are the elements of the vector $\mathbf{f}$. In general, the invertability of the climatological covariance C_∞ is not guaranteed for rank-1 forcing. The climatological covariance C_∞ is invertible for 2×2 diagonal dynamics when the entries of $\mathbf{f}$ are nonzero and the eigenvalues of the dynamics are distinct. For rank-1 forcing, the eigenvalues of G_τ are given by $\lambda_k(\mathsf{G}_\tau) = \lambda_k(\mathsf{E}_\tau \mathsf{E}_\infty^{-1})$ and remarkably, are independent of the forcing $\mathbf{f}$. A direct calculation gives that the predictability measure (23) of the system with rank-1 forcing is

$$1 - \frac{1}{2}\,\mathrm{tr}\,\mathsf{G}_\tau = \frac{1}{2}\left(e^{2\,\lambda_1\,t} + e^{2\,\lambda_2\,t}\right) + \frac{2\lambda_1\lambda_2\left(\left(e^{\lambda_1\tau} - e^{\lambda_2\tau}\right)^2 + 2e^{(\lambda_1+\lambda_2)\tau}\,(1-\cos 2\phi)\right)}{4\phi^2 + (\lambda_1-\lambda_2)^2}\,. \tag{30}$$

The result in (30) is valid when the eigenvalues of the dynamics are distinct. The first term on the right-hand side of (30) is the minimum predictability of the system, and the second term is strictly positive. Therefore, rank-1 forcing gives predictability that is *strictly* larger than normal-mode uncorrelated forcing. Figure 1 compares the quantity $1 - \mathrm{tr}\,\mathbf{G}_\tau/2$ for normal-mode uncorrelated and for rank-1 forcing as function of lead-time τ with $\lambda_1 = -1$ and $\lambda_2 = -2$. In this example, rank-1 stochastic forcing is seen to enhance predictability on time-scales comparable to the system e-folding time. Rank-1 forcing also increases predictability as measured by the predictive information R_τ and the relative entropy r_τ. In fact, there is no rank-2 forcing that gives more predictability. The optimization problems of maximizing $1 - \mathrm{tr}\,\mathbf{G}_\tau/2$, R_τ and r_τ can be solved in closed form for the 2×2 system and show that rank-1 forcing maximizes predictability. Whether rank-1, or perhaps approximately rank-1, forcing gives maximum predictability in general remains an open question. However, limited numerical experiments do not contradict this conjecture.

5. Discussion

In this work we have considered error growth and predictability in linear stochastic systems with perfect initial conditions and state-independent stochastic forcing. We have focused our study on the impact of stochastic forcing spatial structure on error growth and predictability. Two classes of forcing were considered: forcing that is uncorrelated in normal-mode space and forcing with arbitrary spatial structure. Prediction error growth depends strongly on details of the forcing, even for the special case of normal-mode uncorrelated forcing. Properly chosen forcing structures can produce prediction error growth more efficiently or less efficiently than normal mode forcing. Norm-dependence of prediction error growth measures further complicates the analysis of prediction error growth.

Predictability can be defined in a norm-invariant manner using the *predictive information matrix* (Schneider and Griffies, 1999). Remarkably, predictability of a system with normal-mode

uncorrelated forcing is independent of the forcing amplitudes. Moreover, the predictable patterns are simply the eigenmodes of the dynamics. Predictability of systems where the stochastic forcing has general spatial structure is more difficult to analyze. However, in contrast to the results of prediction error growth analysis where general stochastic forcing can either increase or decrease error growth, general stochastic forcing always increases predictability; normal-mode uncorrelated forcing gives minimum predictability. Therefore, normal mode analysis of the predictability problem gives lower bounds for the system with general forcing structure.

These results can also be interpreted in the framework where eigenvectors of the dynamics are allowed to vary and the forcing is fixed as homogeneous and spatially uncorrelated. Homogeneous and spatially uncorrelated forcing is uncorrelated in normal-mode space when the dynamics is normal. Therefore, for homogeneous and spatially uncorrelated stochastic forcing, a system with nonnormal dynamics is more predictable than a system with the same eigenvalues but normal dynamics.

Our results are in the form of lower bounds for predictability. The fact that the lower bound of the predictability depends only the real part of the eigenvalues implies that if the forcing is uncorrelated in the normal mode space, then the predictability does not depend on the oscillatory behavior of the system. An topic for future research is the formulation of *upper* bounds for predictability. We expect such upper bounds to depend only on the eigenvalues of the dynamics, as in the simple example. Another issue is the characterization of forcing structures that give maximum predictability as in Chang et al. (2002) where the forcing producing maximum predictability for norm-dependent error variance predictability measures was found.

Acknowledgement The authors thank Tim DelSole (COLA) for his comments, suggestions and corrections. PC is grateful to IRI for its hospitality and generous support during his sabbatical leave when this research was initiated. PC also acknowledges the supports by the NSF grant ATM-99007625 and the NOAA grant NA16GP1572.

Appendix

The necessary and sufficient condition relating the eigenvalues $\lambda_k(\mathbf{W})$ and singular values $\sigma_k(\mathbf{W})$ of an invertable matrix $\mathbf{W}$ is the inequality

$$\sum_{k=1}^{j} \log \sigma_k(\mathbf{W}) \geq \sum_{k=1}^{j} \log |\lambda_k(\mathbf{W})| \,, \tag{31}$$

for $j = 1, 2, \ldots, n$ with equality for $j = n$ (Marshall and Olkin, 1979). The sequence $\{\log \sigma_k(\mathbf{W})\}$ is said to *majorize* the sequence $\{\log |\lambda_k(\mathbf{W})|\}$. The following classical result of Weyl identifies functions that preserve majorization.

Theorem 1 (Weyl (1949); Chapter 5, A.2.a of Marshall and Olkin (1979)**).** *Suppose* $\mathbf{W}$ *is an invertible* $n \times n$ *matrix with eigenvalues* $\lambda_k(\mathbf{W})$ *and singular values* $\sigma_k(\mathbf{W})$*. If* $h(e^x)$ *is an increasing convex function then,*

$$\sum_{k=1}^{j} h(\sigma_k(\mathbf{W})) \geq \sum_{k=1}^{j} h(|\lambda_k(\mathbf{W})|) \,, \tag{32}$$

for $j = 1, 2, \ldots, n$*.*

The following lemmas are consequences of Theorem 1 with specific choices of the function h. Lemma 1 follows from taking $h(x) = x^s$ and from $h(e^x) = e^{sx}$ being an increasing convex function for $s > 0$.

Lemma 1. *Suppose* $\mathbf{W}$ *is an* $n \times n$ *matrix with with eigenvalues* $\lambda_k(\mathbf{W})$ *and singular values* $\sigma_k(\mathbf{W})$*. For* $s > 0$*,*

$$\sum_{k=1}^{j} \sigma_k^s(\mathbf{W}) \geq \sum_{k=1}^{j} |\lambda_k(\mathbf{W})|^s \,, \quad j = 1, 2, \ldots, n \,. \tag{33}$$

The function $h(e^x)$ is a convex increasing function for $-\infty \leq x \leq 0$ when h is an increasing convex function on the interval $(0, 1)$. The following lemma is a consequence of $-\log(1 - x^2)$ and $-(\log(1 - x^2) + x^2)$ being convex increasing functions on the interval $(0, 1)$.

Lemma 2. *Under the assumptions of Theorem 1, and* $\sigma_1(\mathbf{W}) < 1$

$$-\sum_{k=1}^{j} \log\left(1 - |\sigma_k(\mathbf{W})|^2\right) \geq -\sum_{k=1}^{k} \log\left(1 - \lambda_k^2(\mathbf{W})\right) , \tag{34}$$

and

$$-\sum_{k=1}^{j} \log\left(1 - |\lambda_k(\mathbf{W})|^2\right) + |\lambda_k(\mathbf{W})|^2 \geq -\sum_{k=1}^{j} \log\left(1 - \sigma_k^2(\mathbf{W})\right) + |\sigma_k(\mathbf{W})|^2 , \tag{35}$$

for $j = 1, 2, \ldots, n$.

References

Chang, P., Saravanan, R., DelSole, T., and Wang, F. 2002. Predictability of Linear Coupled Systems. Part I: Theoretical Analyses. *J. Climate*. submitted.

Farrell, B. F. and Ioannou, P. J. 1996. Generalized Stability Theory. Part I: Autonomous Operators. *J. Atmos. Sci.*, **53**, 2025–2040.

Hasselmann, K. 1976. Stochastic climate models I. Theory. *Tellus*, **28**, 473–485.

Kleeman, R. 2001. Measuring dynamical prediction utility using relative entropy. *J. Atmos. Sci.* submitted.

Kleeman, R. and Moore, A. M. 1997. A Theory for the Limitation of ENSO Predictability Due to Stochastic Atmospheric Transients. *J. Atmos. Sci.*, **54**, 753–767.

Lorenz, E. N. 1969. The predictability of a flow which possesses many scales of motion. *Tellus*, **21**, 289–307.

Marshall, A. W. and Olkin, I. *Inequalities: Theory of Majorization and its applications*, volume 143 of *Mathematics in Sciences and Engineering*. Academic Press, New York, New York, 1979.

Penland, C. and Matrosova, L. 1994. A Balance Condition for Stochastic Numerical Models with Application to the El Niño-Southern Oscillation. *J. Climate*, **7**, 1352–1372.

Penland, C. and Matrosova, L. 1998. Prediction of Tropical Atlantic Sea Surface Temperatures Using Linear Inverse Modeling. *J. Climate*, **11**, 483–496.

Schneider, T. and Griffies, S. 1999. A Conceptual Framework for Predictability Studies. *J. Climate*, **12**, 3133–3155.

Tippett, M. K. and Cohn, S. E. 2001. Adjoints and low-rank covariance representation. *Nonlinear Processes in Geophysics*, **8**, 331–340.

Weyl, H. 1949. Inequalities between two kinds of eigenvalues of a linear transformation. *Proc. Nat. Acad. Sci U.S.A.*, **35**, 408–411.

Whitaker, J. S. and Sardeshmukh, P. D. 1998. A Linear Theory of Extratropical Synoptic Eddy Statistics. *J. Atmos. Sci.*, **55**, 237–258.

Winkler, C. R., Newman, M., and Saradeshmukh, P. D. 2001. A Linear Model of Wintertime Low-Frequency Variability. Part I: Formulation and Forecast Skill. *J. Climate*, **14**, 4474–4494.