Modal and nonmodal linear stability of electromagnetic drift-waves*

Michael K. Tippett†, Suzana J. Camargo, Iberê L. Caldas‡
Universidade Estadual Paulista – Unesp, Guaratinguetá, SP, Brazil

Abstract

The modal growth rate of a linearized electromagnetic drift-wave model is zero in both the hydrodynamic and adiabatic limits. However, the linear system is nonnormal in that its eigenvectors are not orthogonal with respect to the energy inner product. Nonnormal linear systems are only weakly constrained by their spectral properties. Previous work with an electrostatic drift-wave model found that modal growth and growth rates are poor estimates of the general nonmodal behavior in the hydrodynamic limit. Similar behavior is seen in the hydrodynamic regime with the addition of magnetic fluctuations. In the adiabatic regime the maximum instantaneous growth rate is orders of magnitude larger than the modal growth rate.

INTRODUCTION

The transport of energy and particles in magnetically confined plasmas is not well understood. Turbulent fluctuations are an often proposed mechanism to explain anomalously large observed transport[1]. Drift-waves, the result of spatial gradients in the plasma, are believed to play an important role in the turbulent dynamics, especially in the cool plasma edge.

Several studies have examined the modal behavior of linear drift-wave dynamics and compared it with the behavior seen in direct numerical simulations of the full nonlinear dynamics[2]. However, modal analysis gives only a partial picture as the linear models are nonnormal i.e., their eigenmodes are not complete and orthogonal[3]. The temporal and spatial scales of nonmodal instabilities in an electrostatic drift-wave model were found to be quite different than those of the modal instabilities[4]. Here the usual eigenvalue analysis is complemented with other mathematical techniques and used to investigate the linear behavior of an electromagnetic drift-wave model.

MODEL

This model generalizes the electrostatic Hasegawa-Wakatani drift-wave model to include magnetic fluctuations[2,5]. The dimensionless equations for the potential ϕ, density n and magnetic fluctuations ψ are:

$$\frac{d_E}{dt}\nabla_\perp^2\phi = -\frac{1}{\beta}\nabla_\parallel\nabla_\perp^2\psi - \nu\nabla_\perp^6\phi\,, \tag{1}$$

$$\frac{d_E}{dt}n = -\frac{\partial\phi}{\partial y} - \frac{1}{\beta}\nabla_\parallel\nabla_\perp^2\psi - \nu\nabla_\perp^4 n\,, \tag{2}$$

$$\frac{\partial\psi}{\partial t} = -\frac{\partial\psi}{\partial y} + \nabla_\parallel(n-\phi) + \frac{1}{\beta D}\nabla_\perp^2\psi - \nu\nabla_\perp^4\psi\,. \tag{3}$$

*This work was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico Grant 381737/97-73 and by Fundação de Amparo à Pesquisa do Estado de São Paulo Grant 96/5388-0.
†Centro de Previsão do Tempo e Estudos Climáticos, INPE, SP, Brazil
‡Instituto de Física, Universidade de São Paulo, São Paulo, SP, Brazil

The character of the system is determined by the the electromagnetic parameter β and the adiabaticity parameter D[2]. Values of $\beta \ll 1$ and $\beta \gg 1$ correspond respectively to the electrostatic and electromagnetic regimes; values of $D \ll 1$ and $D \gg 1$ correspond respectively to the hydrodynamic and adiabatic regimes. Typical parameter values used in investigations of magnetically confined fusion plasmas are $\beta \sim 1$ and $D \sim 10$[2].

The linear model comes from taking

$$\frac{d_E}{dt} \equiv \frac{\partial}{\partial t}\,, \qquad \nabla_{\parallel} \equiv \frac{\partial}{\partial z}\,. \tag{4}$$

SPECTRA, PSEUDOSPECTRA AND NUMERICAL RANGE

Assuming a spatial dependence of the form $\exp i(\mathbf{k}\cdot\mathbf{r})$ with $\mathbf{k} = (k_x, k_y, k_z)$ and $\mathbf{r} = (x, y, z)$, leads to the evolution equation

$$\frac{d}{dt}u = Au\,, \tag{5}$$

where A is the infinite block-diagonal matrix with diagonal blocks

$$A_{\mathbf{k}} = \begin{bmatrix} -\nu k_{\perp}^4 & 0 & -ik_z/\beta \\ -ik_y & -\nu k_{\perp}^4 & ik_z k_{\perp}^2/\beta \\ -ik_z & ik_z & -ik_y - k_{\perp}^2/\beta D - \nu k_{\perp}^4 \end{bmatrix}\,; \tag{6}$$

correspondingly, $u_{\mathbf{k}} = \begin{bmatrix}\phi_{\mathbf{k}} & n_{\mathbf{k}} & \psi_{\mathbf{k}}\end{bmatrix}^T$. In this work we consider bidimensional oscillations with $k_z = 1$ and $k_{\perp}^2 = k_x^2 + k_y^2$.

Constraining the time dependence to have the form $\exp(\lambda t)$ leads to the eigenvalue problem $(\lambda I - A)u = 0$. The maximum modal growth rate γ depends on how far the spectrum of A enters the right-half of the complex plane,

$$\gamma = \max_{\lambda \in \Lambda(A)} \operatorname{Re}\lambda\,. \tag{7}$$

Here $\Lambda(A)$ is the spectrum of A consisting of the values of λ such that $(\lambda I - A)$ is singular. Figure 1(a) shows the dependence of γ on the adiabaticity constant D for $\beta = 5$. The first 64 harmonics in x and y are retained in the calculations (64^2 Fourier coefficients) and the length-scales are as in[2]. The dissipation coefficient is $\nu = 7 \times 10^{-4}$. The modal growth rate has its maximum around $D = 0.1$ and goes to zero in both the hydrodynamic and adiabatic limits.

If A were normal, the modal growth rate γ would give strong conditions on the global and instantaneous behavior of the solution. Namely, one could conclude that for all $t \geq 0$ and for any initial condition $u(0)$ with unit initial energy ($\|u(0)\| = 1$)

$$\|u(t)\| \leq e^{\gamma t}\,, \quad \text{and} \quad \max \frac{1}{2}\frac{d}{dt}\|u(0)\| \leq \gamma \tag{8}$$

with the upper limits being achieved by taking $u(0)$ to be the eigenvector of A whose eigenvalue has real part γ.

However, for nonnormal dynamics the strongest statements that generally apply are

$$\|u(t)\| \leq Ce^{\gamma t}\,, \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{2}\frac{1}{t}\ln\|u(t)\| = \gamma\,; \tag{9}$$

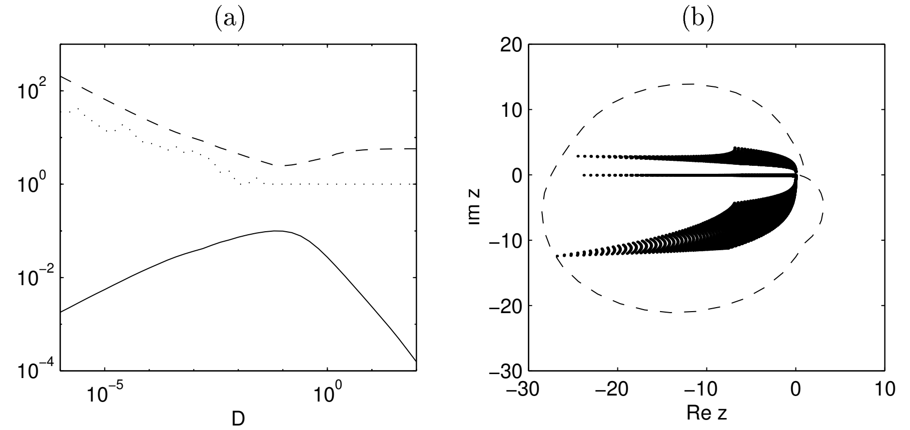

Figure. 1. Panel (a) shows γ (solid line), a Monte Carlo estimate of C_α (dotted line) and C as functions of the adiabaticity constant D for $\beta = 5$. Panel (b) shows the eigenvalues (dots) and the boundary of the numerical range of A (dashed line) for $\beta = 5$ and $C = 10$.

C is a constant greater or equal to unity. The relations in Eq. 9 are considerably weaker than those in Eq. 8. The constant C is not known from the eigenvalue analysis and the modal growth rate γ only limits the long-time growth rate. Therefore, the linear behavior of a nonnormal system is only weakly determined by the properties of its spectrum.

Information about the constant C can be found from the *pseudospectrum* of A[7]. The extension of the pseudospectrum into the right half-plane is measured by the ϵ-*pseudospectral abscissa* $\alpha(\epsilon)$ defined by

$$\alpha(\epsilon) = \max_{z\in\Lambda_\epsilon(A)} \operatorname{Re} z\,. \tag{10}$$

Here $\Lambda_\epsilon(A)$ is the ϵ-pseudospectrum of A; a complex number z is in $\Lambda_\epsilon(A)$ if z is an eigenvalue of $(A+E)$ for $\|E\| \leq \epsilon$. For $\epsilon = 0$ the modal growth rate is recovered, $\gamma = \alpha(0)$. A useful result is that the constant $C = 1$ if and only if $\alpha(\epsilon) \leq \alpha(0) + \epsilon$ for all positive ϵ, i.e., if a perturbation of size ϵ to A moves its eigenvalues by at most a distance ϵ further into the real half-plane[7]. Secondly, it can be shown that[7]

$$C \geq C_\alpha \equiv \max_{\epsilon>0} \frac{\alpha(\epsilon) - \alpha(0)}{\epsilon}\,. \tag{11}$$

Figure 1(a) shows a Monte Carlo estimate of C_α and the exact value of C as a function of D for $\beta = 5$. For $D \ll 1$, C_α and C are considerably greater than unity, indicating the presence of enhanced nonmodal growth. Similar behavior for $D \ll 1$ was seen in an electrostatic (i.e., $\beta \ll 1$) model[4]. However, in the electrostatic model considered in that work, as D increases nonmodal enhanced growth disappears and $C_\alpha \approx 1$ for $D \geq 1$. Here, including moderately strong magnetic fluctuations leads to $C > 1$ even for large values of the adiabaticity constant D. For the range of values $D \geq 10^{-2}$ the Monte Carlo estimate gives $C_\alpha \approx 1$. More detailed calculations show that indeed $C_\alpha > 1$, in agreement with the theory.

The value of C does not give information about the instantaneous rate of change of energy and the corresponding time-scales of the system. The rate of change of the energy is given by

$$\frac{1}{2}\frac{d}{dt}\|u(t)\| = \frac{1}{2}\frac{d}{dt}\langle u(t), u(t)\rangle = \frac{1}{2}\langle u(t), Au(t)\rangle + \frac{1}{2}\langle Au(t), u(t)\rangle\,; \tag{12}$$

$\langle\cdot,\cdot\rangle$ is the energy inner-product. The maximum rate of change of the energy is given by the *numerical abscissa* ω defined by

$$\omega = \max_{z\in W(A)} \operatorname{Re} z\,. \tag{13}$$

Here $W(A)$ is the *numerical range* of A consisting of all complex numbers z such that for some unit vector u, $z = \langle u, Au\rangle$. If A is normal, then $\omega = \gamma$. The eigenvalues of A and the boundary of $W(A)$ are shown in Fig. 1(b) for $\beta = 5$ and $C = 10$. The time-scale suggested by the maximum modal growth rate $\gamma = 0.0024$ is orders of magnitude different from that corresponding to the maximum rate of change of energy $\omega = 3.1$. A simple calculation shows that Eq. 9 implies that the maximum growth rate ω can be maintained for a time no longer than 0.54. Details of how the maximum growth rate ω relaxes to the long-time modal growth rate γ can be obtained by calculating finite-time growth rates as was done for the electrostatic model in[4]

CONCLUSIONS

Nonmodal analysis of the linear behavior of an electromagnetic drift-wave model reveals features not evident from eigenvalue or modal analysis. Examination of the pseudospectrum shows that while the modal growth rate goes to zero in the hydrodynamic limit, nonmodal growth is still present. The pseudospectrum was not as successful in diagnosing nonmodal properties in the adiabatic regime. Calculation of the numerical abscissa showed that in the adiabatic regime the instantaneous rate of the change of the energy can be many orders of magnitude larger than the modal growth rate.

REFERENCES

1. A. J. Wootton, B. A. Carreras, H. Matsumoto, K. McGuire, W. A. Peebles, Ch. P. Ritz, P. W. Terry and S. J. Zweben, "Fluctuations and anomalous transport in tokamaks," Phys. Fluids B, **2**, 2879 (1990).
2. S. J. Camargo, B. D. Scott, and D. Biskamp, "The influence of magnetic fluctuations on collisional drift-wave turbulence," Phys. Plasmas, **3**, 3912 (1996).
3. L. N. Trefethen, A. E. Trefethen, and S. C. Reddy, "Hydrodynamic stability without eigenvalues," Science, **261**, 584 (1993).
4. S. J. Camargo, M. K. Tippett, and I. Caldas, "Nonmodal energetics of resistive drift-waves." Technical Report P-1296, IFUSP (Instituto de Física da Universidade de São Paulo), 1998. to appear in Phys. Rev. E.
5. A. Hasegawa and M. Wakatani, "Plasma Edge Turbulence," Phys. Rev. Lett., **50**, 682, (1985).
6. A. Hasegawa and M. Wakatani, "Finite-larmor-radius magnetohydrodynamic equations for microturbulence" Phys. Fluids, **26**, 2770, (1983).
7. L. N. Trefethen, "Pseudospectra of linear operators," SIAM Review, **39**, 383 (1997).