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Bounds for Solutions of the Discrete Algebraic Lyapunov Equation

Michael K. Tippett and Dan Marchesin

***Abstract*—A family of sharp, arbitrarily tight upper and lower matrix bounds for solutions of the discrete algebraic Lyapunov are presented. The lower bounds are tighter than previously known ones. Unlike the majority of previously known upper bounds, those derived here have no restrictions on their applicability. Upper and lower bounds for individual eigenvalues and for the trace of the solution are found using the new matrix bounds. Sharp trace bounds not derivable from the matrix bounds are also presented.**

***Index Terms*—Covariance matrices, Lyapunov matrix equations, matrix bounds.**

I. Introduction

The discrete algebraic Lyapunov equation (DALE) is

$$P = A^T PA + Q, \qquad A, Q \in R^{n\times n}, \qquad Q = Q^T > 0 \tag{1}$$

where all the eigenvalues of A lie inside the unit circle, $(^T)$ and (>0) denote transpose and positive definiteness, respectively, and $P = P^T > 0$ is the solution. Bounds for solutions of the DALE are often in the form of *eigenvalue bounds*, that is bounds for single eigenvalues of P, bounds for the trace of P, or bounds for the determinant of P. A more general type of bound is a *matrix bound*, such as

$$P \le B, \qquad B = B^T \in R^{n\times n} \tag{2}$$

where the notation $P \le B$ means that the matrix $B - P$ is positive semidefinite. If one has matrix bounds, one may easily derive eigenvalue bounds.

Our particular motivation for seeking bounds for the solution of the DALE comes from the application of the Kalman filter to the problem of assimilating atmospheric data (e.g., [1]). With some simplifying assumptions, the error covariance of the estimate of the state of the atmosphere satisfies the equation in (1) with the appropriate choice of A and Q. For this application, the DALE has two distinguishing properties. First, the system comes from the discretization of a three-dimensional continuum problem; the dimension n of the matrices is large, typically of the order 10^6. Since direct treatment of (1) is impractical, estimates for the solution of the DALE are valuable and can be used, for example, to investigate the dependence of P on A and Q and to develop approximate methods. Second, in atmospheric dynamics, as in fluid dynamics, an important feature of the dynamics is nonmodal growth due to nonnormality [2], [3]. When such nonmodal growth is present, A is nonnormal and has singular values greater than one. The majority of previously known

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upper bounds for the solution of the DALE are inapplicable when the singular values of A are greater than one (e.g., [4]). This lack of upper bounds may indicate a lack of theoretical understanding of the behavior of solutions of the DALE. The upper bounds derived here, applicable with no restrictions on the singular values of A, are an expansion of the theoretical understanding of the DALE.

First, in Theorem 1 we recall that the solution of the DALE has a series representation. Then, we use that series representation to derive Theorem 2, a family of sharp, arbitrarily tight matrix bounds. Naturally, the amount of work required to calculate a very tight bound approaches that of computing the solution. We show that the new lower bounds are tighter than bounds recently presented in [5]. The new upper bounds do not become unbounded when the singular values of A approach one and are applicable when the singular values of A are greater than one. The bounds also show that solutions of the DALE depend on both the eigenvalues of A and on the sensitivity of these eigenvalues to perturbations. The eigenvalues of nonnormal matrices may be extremely sensitive to perturbations.

With the new matrix bounds we derive Corollaries 1 and 2, upper and lower bounds for the individual eigenvalues of P and for the trace of P. Finally in Theorem 3, we derive sharp upper and lower bounds for the constant of proportionality that relates the trace of P with the trace of Q.

For $X \in R^{n\times n}$, we use the notation $\lambda_i(X)$ and $\sigma_i(X)$ to denote the ith eigenvalue and singular value of the matrix X, where $|\lambda_i(X)|$, $i = 1, \cdots, n$, and $\sigma_i(X)$, $i = 1, \cdots, n$ are in nonincreasing order. The norm $\|\cdot\|_2$ is the usual two-norm with $\|X\|_2 = \sigma_1(X)$. We will use the standard results for eigenvalues of symmetric matrices [6]

$$\sigma_n^2(Y)\lambda_i(X) \le \lambda_i(Y^TXY) \le \sigma_1^2(Y)\lambda_i(X) \tag{3}$$

for $X = X^T$, $Y \in R^{n\times n}$, and

$$\lambda_i(X) + \lambda_n(Y) \le \lambda_i(X+Y) \le \lambda_i(X) + \lambda_1(Y) \tag{4}$$

for $X = X^T$, $Y = Y^T \in R^{n\times n}$. Standard properties of the trace operator are (see, e.g., [7])

$$\mathrm{tr}(X+Y) = \mathrm{tr}(X) + \mathrm{tr}(Y) \tag{5}$$

for $X, Y \in R^{n\times n}$ and

$$\mathrm{tr}(XY) = \mathrm{tr}(YX) \tag{6}$$

for $X = \in R^{p\times n}, \quad Y = \in R^{n\times p}$.

II. Results

Definition 1: For $A \in R^{n\times n}$ with eigenvalues inside the unit circle and for any integer $m \ge 0$, define $P_m = P_m^T$ by $P_0 = 0$ and

$$P_m = \sum_{k=0}^{m-1} (A^T)^k Q A^k, \qquad m \ge 1. \tag{7}$$

Theorem 1: The solution P of (1) is

$$P = \lim_{m\to\infty} P_m = \sum_{k=0}^{\infty} (A^T)^k Q A^k. \tag{8}$$

Proof: See, e.g., [8]. □

Definition 2: For $A \in R^{n\times n}$ with eigenvalues inside the unit circle and for any integer $m \ge 0$, define $H_m = H_m^T$ by

$$H_m = \sum_{k=m}^{\infty} (A^T)^k A^k. \tag{9}$$

The terms of the series in (9) can be bounded using [7]

$$\sigma_1^2(A^k) \le (1+\theta)^{2n-2}\left(|\lambda_1(A)| + \frac{\|N\|_F}{1+\theta}\right)^{2k} \tag{10}$$

for any $\theta \ge 0$, where $A = Z^T(D+N)Z$ is the Schur decomposition of A. Recall that if A is normal $N \equiv 0$. Since $|\lambda_1(A)| < 1$ one may choose a finite θ such that

$$\mu \equiv |\lambda_1(A)| + \frac{\|N\|_F}{1+\theta} < 1. \tag{11}$$

Thus

$$\|H_m\|_2 \le \sum_{k=m}^{\infty} \sigma_1^2(A^k) \le (1+\theta)^{2n-2}\frac{\mu^{2m}}{1-\mu^2} \tag{12}$$

showing the series in (9) converges and $\lim_{m\to\infty} H_m = 0$. Also

$$H_m - H_{m+1} = (A^T)^m A^m \ge 0 \tag{13}$$

implies that

$$H_0 \ge H_1 \ge H_2 \cdots \ge 0. \tag{14}$$

Theorem 2: If P is a solution of (1), then for all integers $m \ge 0$

$$\lambda_n(Q)H_m + P_m \le P \le \lambda_1(Q)H_m + P_m. \tag{15}$$

Proof: Since $Q = Q^T > 0$, Q can be expressed as

$$Q = \sum_{i=1}^{n} \lambda_i(Q) w_i w_i^T \tag{16}$$

where $\lambda_i(Q) > 0$ and the w_i are orthogonal. For any $x \in R^n$ it can be seen that

$$\begin{aligned} x^TPx &= x^TP_mx + \sum_{k=m}^{\infty} x^T(A^T)^kQA^kx \\ &= x^TP_mx + \sum_{k=m}^{\infty}\sum_{i=0}^{n} \lambda_i(Q)x^T(A^T)^k w_i w_i^T A^k x \\ &\ge x^TP_mx + \sum_{k=m}^{\infty}\sum_{i=0}^{n} \lambda_n(Q)x^T(A^T)^k w_i w_i^T A^k x \\ &= x^TP_mx + \lambda_n(Q)x^TH_mx. \end{aligned} \tag{17}$$

Similarly

$$\begin{aligned} x^TPx &\le x^TP_mx + \sum_{k=m}^{\infty}\sum_{i=0}^{n} \lambda_1(Q)x^T(A^T)^k w_i w_i^T A^k x \\ &= x^TP_mx + \lambda_1(Q)x^TH_mx. \end{aligned} \tag{18}$$

□

Applying (4) to (15) gives the following.

Corollary 1: If P is a solution of (1), then

$$\lambda_n(Q)\lambda_n(H_m) + \lambda_i(P_m) \le \lambda_i(P) \le \lambda_1(Q)\lambda_1(H_m) + \lambda_i(P_m). \tag{19}$$

Applying (5) to (15) gives the following.

Corollary 2: If P is a solution of (1) then

$$\lambda_n(Q)\,\mathrm{tr}\,H_m + \mathrm{tr}\,P_m \le \mathrm{tr}\,P \le \lambda_1(Q)\,\mathrm{tr}\,H_m + \mathrm{tr}\,P_m. \tag{20}$$

Theorem 3 and its proof are analogous to results in [9] for the continuous Lyapunov equation.

Theorem 3: If P is a solution of (1) then

$$\lambda_n(H_{0T})\,\mathrm{tr}\,Q \le \mathrm{tr}\,P \le \lambda_1(H_{0T})\,\mathrm{tr}\,Q \tag{21}$$

where

$$H_{0T} = \sum_{k=0}^{\infty} A^k(A^T)^k. \tag{22}$$

Proof: Since $Q = Q^T > 0$, Q can be expressed as

$$Q = \sum_{i=1}^{n} \lambda_i(Q) w_i w_i^T \tag{23}$$

where w_i are the orthogonal eigenvectors of Q and $\lambda_i(Q) > 0$. Hence

$$\begin{aligned} \operatorname{tr}(P) &= \sum_{i=1}^{n}\sum_{k=0}^{\infty} \lambda_i(Q) \operatorname{tr}\big((A^T)^k w_i w_i^T A^k\big) \\ &= \sum_{i=1}^{n}\sum_{k=0}^{\infty} \lambda_i(Q) \operatorname{tr}\big(w_i^T A^k (A^T)^k w_i\big) \\ &= \sum_{i=1}^{n} \lambda_i(Q) w_i^T H_{0T} w_i \end{aligned} \tag{24}$$

where we use the property of the trace operator given in (6). However

$$\lambda_n(H_{0T}) \le w_i^T H_{0T} w_i \le \lambda_1(H_{0T}) \tag{25}$$

and

$$\operatorname{tr} Q = \sum_{i=1}^{n} \lambda_i(Q). \tag{26}$$

□

III. Remarks and Comparisons to Existing Bounds

Remark 1: The bounds in Theorem 2 are sharp in the sense that for any A there is a Q such that the bounds are achieved; for $Q = I$, the solution of (1) is $P = H_0$ and the family of inequalities in Theorem 2 collapses to a single equality. They are also sharp in the sense that for any Q there is an A such that the bounds are achieved. For Q as in (16), the lower bound of Theorem 2 is achieved if $A = \lambda w_n w_n^T$ and the upper bound if $A = \lambda w_1 w_1^T$ with $|\lambda| < 1$. The bounds in Theorem 2 are arbitrarily tight since for large enough m, both $\|H_m\|_2$ and $\|P - P_m\|_2$ are arbitrarily small.

Remark 2: To use these new bounds requires calculating the matrices H_m. If A is normal, then A can be written as $A = V^T D V$ where V is orthogonal and $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$. In this case, the form of H_m is quite explicit, $H_m = V^T D' V$; $D' = \operatorname{diag}(\gamma_1(m), \cdots, \gamma_n(m))$ with

$$\gamma_i(m) = \frac{|\lambda_i(A)|^{2m}}{1 - |\lambda_i(A)|^2} \tag{27}$$

and the size of H_m depends only on m and the closeness of the eigenvalues of A to the unit circle.

When A is nonnormal the cost of calculating H_m is the same as solving the DALE, which for large systems may be impractical. However, truncating the series in (9) gives lower bounds such as

$$H_0 \ge \sum_{k=0}^{L} (A^T)^k A^k \tag{28}$$

for any integer $L \ge 1$. Such a truncation is useful only in the lower bounds; the cost of calculating the upper bounds remains high if the series in (9) converges slowly. For $L = 1$, (28) shows that the eigenvalues of H_0 are bounded from below by

$$\lambda_i(H_0) \ge 1 + \sigma_i^2(A) \tag{29}$$

demonstrating that in contrast to when A is normal, when A is nonnormal H_0 can be large even when the eigenvalues of A are not near the unit circle. The cost of calculating the leading eigenvalues and eigenvectors of the truncated series in (28) is roughly a factor of L times that of calculating the leading singular values and vectors of A. Calculation of the leading singular values and vectors of A may be practical using iterative Lanczos methods even when n is large (e.g., [7]). For example, if A comes from the discretization of a partial differential equation then the cost of applying A and A^T to a vector may not be too great and iterative methods are practical (e.g., [10]).

Remark 3: In [5] the lower matrix bound

$$P \ge \frac{\lambda_n(Q)}{1 - \sigma_n(A)} A^T A + Q \tag{30}$$

was presented and used to derive new eigenvalue bounds tighter than the majority of previously existing results. The lower matrix bound given by Theorem 2 for $m = 1$ is

$$P \ge \lambda_n(Q) H_1 + Q. \tag{31}$$

The bound in (31) is as tight or tighter than the one in (30) when

$$H_2 \ge \frac{\sigma_n^2(A)}{1 - \sigma_n^2(A)} A^T A. \tag{32}$$

Clearly

$$\frac{\sigma_n^2(A)}{1 - \sigma_n^2(A)} v_i^T A^T A v_i = \frac{\sigma_n^2(A)\sigma_i^2(A)}{1 - \sigma_n^2(A)} \tag{33}$$

where v_i is the ith right singular vector of A. However

$$\begin{aligned} v_i^T H_2 v_i &= \sum_{k=2}^{\infty} v_i^T (A^T)^k A^k v_i \\ &= \sigma_i^2(A) \sum_{k=1}^{\infty} u_i^T (A^T)^k A^k u_i \\ &\ge \sigma_i^2(A) \sum_{k=1}^{\infty} \sigma_n^{2k}(A) \\ &= \frac{\sigma_n^2(A)\sigma_i^2(A)}{1 - \sigma_n^2(A)} \end{aligned} \tag{34}$$

where u_i is the ith left singular vector of A and by definition $Av_i = \sigma_i(A) u_i$. Since the vectors v_i span R^n, the lower bounds in Theorem 2 for $m \ge 1$ are tighter than the one in (30).

Remark 4: In Remark 3 it was shown rigorously that the new lower bounds are tighter than the one in (30). We now show a few examples to illustrate this tightness. Roughly speaking, the lower bounds presented here are tighter than that in (30) because the latter uses only the first term of the series in (9). First, we consider the example presented in [5, Remark 5]

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{35}$$

The exact solution is

$$P = \begin{bmatrix} (1-\lambda^2)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \tag{36}$$

and the lower bound from (30) is

$$P \ge \begin{bmatrix} 1+\lambda^2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{37}$$

For λ near zero this is a good bound because $(1-\lambda^2)^{-1} \approx 1 + \lambda^2$. However, as mentioned in [5], it becomes an increasingly conservative lower bound as λ approaches one. Since for $Q = I$ the bounds in Theorem 2 become a single equality, $P = H_0$, it is more meaningful to examine what happens when the approximation in (28) is used. For this example (28) becomes

$$P = H_0 \ge \begin{bmatrix} \sum_{i=0}^{L} \lambda^{2i} & 0 \\ 0 & 1 \end{bmatrix} \tag{38}$$

which is a better bound than (37) when $L > 2$.

As a second example, take A to be the $n \times n$ matrix with ones on the superdiagonal and zero elsewhere and take $Q = I$. For this example $\|P\|_2 = n$, while the bound from (30) is

$$\|P\|_2 \geq \|A^T A + I\|_2 = 2. \tag{39}$$

The bound obtained using the truncated series in (28) is $\|P\|_2 \geq L$ for $L \leq n$ and $\|P\|_2 \geq n$ for $L > n$ which, for $L > 2$, is a tighter bound than that in (39).

Remark 5: The upper matrix bound

$$P \leq \frac{\lambda_n(Q)}{1 - \sigma_1(A)} A^T A + Q, \qquad \sigma_1(A) < 1 \tag{40}$$

was derived in [5]. This bound (and eigenvalue bounds derived from it) becomes unbounded as $\sigma_1(A) \to 1$ and is not useful for $\sigma_1(A) > 1$ which may occur when A is nonnormal. An example showing that solutions of (1) need not exhibit any dramatic behavior near $\sigma_1(A) = 1$ is

$$A = \sigma u v^T, \qquad u, v \in R^n, \qquad \|u\|_2 = \|v\|_2 = 1, \qquad \sigma > 0. \tag{41}$$

It is easy to see that

$$\sigma_1(A) = \sigma, \qquad \lambda \equiv \lambda_1(A) = \sigma v^T u. \tag{42}$$

We take $|\lambda|$ to be less than one. For this A, the solution of (1) is

$$P = Q + \frac{\sigma^2}{1 - \lambda^2}(u^T Q u) v v^T. \tag{43}$$

Clearly, in this example for fixed λ the solution is a well-behaved function of σ near $\sigma = 1$. This example also shows the relevance of eigenvalue sensitivity. The sensitivity of an eigenvalue to perturbations is given by its *condition* (see, e.g., [7]). The condition s of a simple eigenvalue is defined in such a way that roughly speaking, a perturbation of order ϵ to a matrix results in a perturbation of the eigenvalue of order ϵ/s. All the eigenvalues of a normal matrix have condition equal to one. Hence, eigenvalue sensitivity is a phenomena associated with nonnormality. Here, the condition of λ is $s = |v^T u|$. The solution P can be written in terms of s as

$$P = Q + \frac{1}{s^2} \frac{\lambda^2}{1 - \lambda^2}(u^T Q u) v v^T. \tag{44}$$

It clear that the solution is potentially unbounded as either $|\lambda| \to 1$ or as $s/\lambda \to 0$.

That the matrix bounds in Theorem 2 and 3 do not become unbounded as $\sigma_1(A) \to 1$ or require that $\sigma_1(A) < 1$ can be seen in the bound for $\|H_m\|_2$ in (12). As in (44) the bound in (12) is related to the nearness of $|\lambda_1(A)|$ to one and to the sensitivity of the eigenvalues of A since

$$1 + \theta > \frac{\|N\|_F}{1 - |\lambda_1(A)|}. \tag{45}$$

The connection between N and eigenvalue sensitivity is seen from the standard result (e.g., [7, Th. 7.2-3]).

Theorem 4: Let $A = Z^T(D + N)Z$ be the Schur decomposition of $A \in R^{n\times n}$. For any $E \in R^{n \times n}$, let η be an eigenvalue of $A + E$ and p be the smallest integer such that $|N^p| = 0$. Then

$$\min_i |\lambda_i(A) - \eta| \leq \max(\phi, \phi^p) \tag{46}$$

where

$$\phi = \|E\|_2 \sum_{k=0}^{p-1} \|N\|_2^k. \tag{47}$$

A second method of bounding $\|H_m\|_2$ can be used if $A = V^{-1}DV$ with $D = \mathrm{diag}(\lambda_1(A), \cdots, \lambda_n(A))$. Then the bound

$$\sigma_1(A^k) \leq \sigma_1(V^{-1})\sigma_1(D)^k \sigma_1(V) = \kappa_2(V)|\lambda_1(A)|^k \tag{48}$$

where $\kappa_2(V) \equiv \|V\|_2 \|V^{-1}\|_2$ is the condition number of the eigenvectors of A, can be used to show

$$\|H_m\|_2 \leq \kappa_2^2(V) \frac{|\lambda_1(A)|^{2m}}{1 - |\lambda_1(A)|^2}. \tag{49}$$

The connection between $\kappa_2(V)$ and eigenvalue sensitivity is apparent, since for η and E as in Theorem 4 one has (e.g., [7, Th. 7.2-2])

$$\min_i |\lambda_i(A) - \eta| \leq \kappa_2(V)\|E\|_2. \tag{50}$$

Remark 6: The bounds in Theorem 3 are sharp in the sense that for any A there is a Q such that the bounds are achieved. Let

$$Q = z z^T. \tag{51}$$

From (24)

$$\mathrm{tr}(P) = z^T H_{0T} z \,\mathrm{tr}\, Q. \tag{52}$$

If z is chosen to be the eigenvector associated with the largest eigenvalue of H_{0T} then

$$\mathrm{tr} P = \lambda_1(H_{0T}) \,\mathrm{tr}\, Q. \tag{53}$$

Similarly, if z is chosen to be the eigenvector associated with the smallest eigenvalue of H_{0T} then

$$\mathrm{tr} P = \lambda_n(H_{0T}) \,\mathrm{tr}\, Q. \tag{54}$$

Finally, if $A = \lambda I$, $|\lambda| < 1$, then for any Q the bounds in Theorem 3 become a single equality.

IV. Conclusion

Here, a family of new, sharp, arbitrarily tight upper and lower matrix bounds for solutions of the DALE was derived. The lower bounds are tighter than previously known ones. For large systems, calculation of the lower bounds using iterative methods may be possible when a direct solution of the DALE is not practical. The new upper bounds, though costly to compute, are of particular interest as they do not require that the singular values of A be less than one. The bounds also suggest that solutions of the DALE depend on both the eigenvalues of A and on the sensitivity of these eigenvalues to perturbations. Using these matrix bounds, bounds for the individual eigenvalues and for the trace of the solution were found. Additionally, sharp upper and lower bounds relating the trace of P and the trace of Q were presented.

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Static Output Feedback Control for Periodically Time-Varying Systems

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***Abstract*—Most control designs for periodically time-varying systems use either full-state feedback or observer-based state feedback. In this paper, it is shown that *static* output feedback is sufficient for the exponential stabilization of a periodical system under both the controllability and observability assumptions. In fact, by incorporating a new generalized hold function in the control design, one is able to arbitrarily shift all the Poincaré exponents of the periodical system. Most importantly, the control signal is guaranteed to be *continuous* in time while the control signal from previous designs may be discontinuous.**

***Index Terms*—Continuous-time system, generalized hold function, periodically time-varying system, Poincaré exponents, static output feedback.**

I. Introduction

An important class of linear time-varying systems in the physical world is the class of periodical systems, in which the system parameters vary periodically. Analysis for such systems has been done thoroughly in the past [1], [2]. One of the most important results is summarized in the Floquet theory, which states that the stability property of a linear periodical system can be determined by n constant numbers called the Poincaré exponents, where n is the dimension of the system. As in the time-invariant case, if all the Poincaré exponents are in the open left-half plane, the periodical system is exponentially stable. If at least one of the Poincaré exponents is in the open right-half plane, the system is unstable.

For the stability synthesis of periodical systems, most control designs are based on the assumption that all the state variables are accessible for measurement. Among these, the earliest approach is the LQ optimal control, in which one solves a periodical Riccati equation to obtain a stabilizing state feedback control [3], [4]. Another approach is the modal control proposed in [5] which can arbitrarily shift only one of the Poincaré exponents of the system. Later, a layer of modal controllers is suggested to shift *all* the Poincaré exponents [6]. Recently, the generalized hold function design, originally developed in [7], is applied to the state feedback control of a periodical system [8]. However, the resultant control signal may have large discontinuities in time. In practice, such large discontinuities are either unacceptable under the actuator constraint or undesirable due to the possible excitation of high-frequency unmodeled dynamics. Even though an attempt has been made to make the control signal continuous, its success is obstructed by a singularity problem [8].

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In this paper, a new design is proposed to avoid discontinuities in the control signal. Furthermore, it is shown that when the periodical system is both controllable and observable, simple *static output* feedback control is sufficient for the arbitrary assignment of all the Poincaré exponents (note that full *state* feedback is required in [8]). The key elements in the new control design are the well-known Floquet transformation [5] and a new generalized hold function design. This paper is arranged as follows. In Section II, the definition of Poincaré exponents for a periodical system is presented. In Section III, a *discontinuous* output feedback control is developed to assign the Poincaré exponents of the closed-loop system, and the control design is further modified in Section IV in order to remove discontinuities in the control signals.

II. Stability Analysis for Periodical Systems

Consider the stability analysis of the following system:

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

where $x(t) \in R^n$ is the state vector and the system matrix $A(t) \in R^{n\times n}$ is T-periodic in the sense that

$$A(t+T) = A(t), \qquad \forall t > 0.$$

In the famous Floquet theory [1], the stability property of (1) is studied through a state transformation into a new coordinate, on which the system matrix becomes time invariant. Such a transformation, called the *Floquet transformation*, is given by

$$z(t) = P(t)x(t), \qquad P(t) = e^{Jt}\Phi^{-1}(t,0) \tag{2}$$

where $\Phi(t,0)$ is the state transition matrix [2] of (1), satisfying

$$\frac{\partial \Phi(t,\tau)}{\partial t} = A(t)\Phi(t,\tau), \qquad \Phi(t,t) = I, \qquad \Phi(\tau,t) = \Phi^{-1}(t,\tau) \tag{3}$$

and J is a constant matrix given by

$$J = \frac{1}{T}\ln \Phi(T,0). \tag{4}$$

From (1)–(3), the periodical system (1) has a *constant* representation in the new coordinate

$$\dot{z}(t) = Jz(t). \tag{5}$$

One can verify (see [2]) that the state transformation matrix $P(t)$ in (2) is also T-periodic and remains uniformly bounded and nonsingular. The stability property of the periodical system (1) can then be inferred from that of the constant system (5). In the literature, the eigenvalues of the constant matrix J in (5) are referred to as the *Poincaré (or characteristic) exponents*

$$\text{P.E.} \triangleq \lambda_i(J) = \frac{1}{T}\ln \lambda_i[\Phi(T,0)] \tag{6}$$

where the second equality results from (4). The condition for exponential stability of (1) is thus

$$\text{Re}[\lambda_i(J)] < 0 \tag{7}$$

or equivalently

$$|\lambda_i[\Phi(T,0)]| < 1$$