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Upper bounds for the solution of the discrete algebraic Lyapunov equation☆

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Abstract

New upper bounds for the solution of the discrete algebraic Lyapunov equation (DALE) $P = APA^{\mathrm{T}} + Q$ are presented. The only restriction on their applicability is that A be stable; there are no restrictions on the singular values of A nor on the diagonalizability of A. The new bounds relate the size of P to the radius of stability of A. The upper bounds are computable when the large dimension of A make direct solution of the DALE impossible. The new bounds are shown to reflect the dependence of P on A better than previously known upper bounds. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The discrete algebraic Lyapunov equation (DALE) is

$$P = APA^{\mathrm{T}} + Q, \qquad A, Q \in \mathbb{R}^{n\times n}, \qquad Q = Q^{\mathrm{T}} > 0, \tag{1}$$

where all the eigenvalues of A lie inside the unit circle; $(^{\mathrm{T}})$ and (>0) denote transpose and positive definiteness, respectively; $P = P^{\mathrm{T}} > 0$ is the solution. The matrix P is the steady-state covariance of a discrete-time, stochastically forced, stable linear system,

$$x_{k+1} = Ax_k + b_{k+1}, \quad x_k, b_k \in \mathbb{R}^n, \tag{2}$$

b_k is a random forcing with $\langle b_j \rangle = 0$ and $\langle b_j b_k \rangle = \delta_{jk} Q$ where $\langle \cdot \rangle$ denotes expectation and δ_{jk} is the Kronecker delta. Defining the covariance of x_k at time-step k by $P_k \equiv \langle x_k x_k^{\mathrm{T}} \rangle$, the steady-state covariance $P = \lim_{k\to\infty} P_k$ is given by the DALE.

There are many known upper and lower bounds for the solution of the DALE that relate properties of A and Q to the size of P (Kwon et al., 1996). A measure of the value of such bounds is whether they (i) are easier to compute than the actual solution P or (ii) offer some theoretical insight. By these measures the usefulness of a particular bound depends on the application. For instance, few bounds would be dramatically cheaper to compute than the actual solution in low-dimensional problems with fixed A and Q. Our interest in the DALE comes from the application of Kalman-filter-based methods to the problem of estimating the state of the atmosphere (Cohn, 1997). Lyapunov equations have also been used in other atmospheric science applications (Penland, 1989; Whitaker and Sardeshmukh, 1998). Distinguishing properties of the matrix A in these applications are (i) its large dimension and (ii) its nonnormality. The immediate consequences of these two points are (i) impossibility of direct solution and (ii) inapplicability of known bounds.

Large dimension is typical of approximations of systems governed by partial differential equations. For realistic atmospheric models, the dimension n is often $O(10^6)$ or larger. Therefore while the deterministic dynamics in Eq. (2) is computable, the storage of P, the multiplication of the product APA^{T} and the direct solution of Eq. (1) are all impossible. Bounds for P would provide valuable practical and theoretical information in this application. Of course, the bounds should be computable even n is large. Iterative (Krylov) methods are valuable tools for

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extracting information from A when n is large (Golub and Van Loan, 1996, Ch. 10). The same iterative methods routinely used to compute leading (or trailing) singular values and vectors of A with $n \geq 10^5$ could be used to evaluate the bounds presented here (Buizza et al., 1997).

The applicability of most known upper bounds for the solution of the DALE is limited when A is nonnormal. A matrix is said to be nonnormal when it does not have a complete set of orthogonal eigenvectors. Typical indications of the nonnormality of A seen in applications are (i) though stable A may have singular values greater than unity and (ii) the eigenvectors of A may be poorly conditioned (Farrell, 1988). Many upper bounds for the solution of the DALE are infinite or inapplicable when the singular values of A are greater than or equal to unity (Kwon et al., 1996). Other upper bounds depend on the conditioning of the eigenvectors of A and are infinite when A does not have a complete set of eigenvectors (Gahinet et al., 1990). Such bounds are inapplicable in many applications and can be "flawed measures" of the behavior of the solution of the DALE, being unbounded when the solution is nicely behaved.

Here, we present upper bounds that are applicable with no restrictions on A and that are unbounded if and only if the solution itself is unbounded. The bounds here also give considerable theoretical insight as to the behavior of the DALE, showing precisely which properties of A determine P. Such information can used to develop and understand approximations for the dominant part of P. In previous work we showed that when A is highly nonnormal and has singular values much larger than unity, P will have some large components (Tippett, 1998). Here we show that the complete information about the DALE is contained not in the spectrum or singular values of A but in its resolvent. Interpretation of the resolvent-based bounds in terms of the pseudospectrum of A provides a link to a substantial and active body of scientific research (Trefethen et al., 1993; Trefethen, 1997).

The paper is organized as follows. The new results are presented in Section 2. We use an integral representation for the solution of the DALE to derive upper bounds for P that depend on the radius of stability of A. In Section 3 we comment on the significance and utility of the results. We show that the new bounds can be understood depending on the pseudospectrum of A. We demonstrate that the new bounds can be used to answer a conjecture made in Gahinet et al. (1990). We present simple examples comparing the new bound to previously known ones and prove that the new upper bounds go to infinity if and only if the solution itself is unbounded. Finally, we apply the new bounds to the discrete approximation of a partial differential equation.

For $X \in \mathbb{R}^{n\times n}$, we use the notation $\lambda_i(X)$ and $\sigma_i(X)$ to denote the ith eigenvalue and singular value of the matrix X, where $|\lambda_i(X)|$, $i = 1, \ldots, n$ and $\sigma_i(X)$, $i = 1, \ldots, n$ are in nonincreasing order. We will use the usual p-norm $\|\cdot\|_p$ and the Frobenius norm $\|\cdot\|_F$. The Frobenius and p-norms for $1 \leq p \leq \infty$ have the following properties for all X and $Y \in \mathbb{R}^{n\times n}$ (Golub and Van Loan, 1996):

$$\|X + Y\| \leq \|X\| + \|Y\|, \tag{3}$$

$$\|XY\| \leq \|X\|\,\|Y\|, \tag{4}$$

$$\|X^{\mathrm{T}}\| = \|X\|. \tag{5}$$

Standard properties of the trace operator are (Golub and Van Loan, 1996):

$$\mathrm{tr}(X + Y) = \mathrm{tr}(X) + \mathrm{tr}(Y), \tag{6}$$

$$\mathrm{tr}(XY) = \mathrm{tr}(YX) \tag{7}$$

for $X, Y \in \mathbb{R}^{n\times n}$. For $X, Y \in \mathbb{R}^{n\times n}$ and symmetric positive semi-definite, there is the upper bound (Gajić and Qureshi, 1995, Eq. (3.19))

$$\mathrm{tr}(XY) \leq \|X\|_2 \,\mathrm{tr}\, Y. \tag{8}$$

2. Results

Definition 1. The radius of stability of A, $r(A)$ is defined by (Mori, 1990)

$$r(A) = \inf_{0 \leq \theta \leq 2\pi} \|R(\mathrm{e}^{\mathrm{i}\theta}, A)\|^{-1}, \tag{9}$$

where $\|\cdot\|$ is a particular matrix norm and the resolvent of A, $R(\lambda, A)$ is defined by

$$R(\lambda, A) = (\lambda I - A)^{-1}. \tag{10}$$

Denote by $r_p(A)$ and $r_F(A)$ the radii obtained by using the p-norm and Frobenius norm respectively in Eq. (9).

Theorem 2. *The solution P of Eq.* (1) *is*

$$P = \frac{1}{2\pi}\int_0^{2\pi} R(\mathrm{e}^{\mathrm{i}\theta}, A)\, QR(\mathrm{e}^{-\mathrm{i}\theta}, A^{\mathrm{T}})\,\mathrm{d}\theta. \tag{11}$$

Proof. From Lancaster (1970), the solution of Eq. (1) is

$$P = \frac{1}{2\pi \mathrm{i}}\int_\Gamma R(\lambda, A) Q (I - \lambda A^{\mathrm{T}})^{-1}\,\mathrm{d}\lambda, \tag{12}$$

where Γ is a curve that encloses the eigenvalues of A. Choosing Γ to be the unit circle and taking $\lambda = \mathrm{e}^{\mathrm{i}\theta}$ gives the integral in Eq. (11).

Theorem 3. *The solution P of Eq.* (1) *satisfies*

$$\|P\| \leq \frac{\|Q\|}{2\pi}\int_0^{2\pi} \|R(\mathrm{e}^{\mathrm{i}\theta}, A)\|^2\,\mathrm{d}\theta \leq \|Q\|\, r^{-2}(A), \tag{13}$$

where $\|\cdot\|$ may be the Frobenius or p-norm, $1 \leq p \leq \infty$.

Proof. From Eqs. (5) and (3)

$$\|P\| \le \frac{1}{2\pi}\int_0^{2\pi} \|R(e^{i\theta}, A) Q R(e^{-i\theta}, A^T)\| \, d\theta . \tag{14}$$

Using Eq. (4) gives

$$\|P\| \le \frac{1}{2\pi}\int_0^{2\pi} \|R(e^{i\theta}, A)\| \, \|Q\| \, \|R(e^{-i\theta}, A^T)\| \, d\theta . \tag{15}$$

Observing that $\|R(e^{i\theta}, A)\| = \|\mathbb{R}(e^{-i\theta}, A^T)\|$ gives the first inequality in Eq. (13). The second inequality follows by noting that for $0 \le \theta \le 2\pi$

$$\|R(e^{i\theta}, A)\|^2 \le r^{-2}(A). \tag{16}$$

Theorem 4. *The solution P of Eq.* (1) *satisfies*

$$\operatorname{tr} P \le \frac{\|Q\|_2}{2\pi}\int_0^{2\pi} \|R(e^{i\theta}, A)\|_F^2 \, d\theta \le \|Q\|_2 \, r_F^{-2}(A) \tag{17}$$

and

$$\operatorname{tr} P \le \frac{\operatorname{tr} Q}{2\pi}\int_0^{2\pi} \|R(e^{i\theta}, A)\|_2^2 \, d\theta \le \operatorname{tr} Q \, r_2^{-2}(A). \tag{18}$$

Proof. Using Eq. (6) gives

$$\operatorname{tr} P = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}(R(e^{i\theta}, A) Q R(e^{-i\theta}, A^T)) \, d\theta . \tag{19}$$

Applying Eq. (7) gives

$$\operatorname{tr} P = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}(Q R(e^{-i\theta}, A^T) R(e^{i\theta}, A)) \, d\theta . \tag{20}$$

Then Eq. (8) can be used with Q playing the role of either X or Y. This results in

$$\operatorname{tr} P \le \frac{\|Q\|_2}{2\pi}\int_0^{2\pi} \operatorname{tr}(R(e^{-i\theta}, A^T) R(e^{i\theta}, A)) \, d\theta \le \|Q\|_2 r_F^{-2}(A) \tag{21}$$

or

$$\operatorname{tr} P \le \frac{\operatorname{tr} Q}{2\pi}\int_0^{2\pi} \|R(e^{-i\theta}, A^T) R(e^{i\theta}, A)\|_2 \, d\theta \le r_2^{-2}(A) \operatorname{tr} Q . \tag{22}$$

By definition $\operatorname{tr}(R(e^{i\theta}, A) R(e^{-i\theta}, A^T)) = \|R(e^{i\theta}, A)\|_F^2$.

3. Remarks

Remark 5. The quantity $r_p(A)$ gives a measure of the distance of A to an unstable matrix since an equivalent definition for $r_p(A)$ is (Golub and Van Loan, 1996, Eq. (2.7.6))

$$r_p(A) = \inf_{0 \le \theta \le 2\pi} \inf_{E_\theta} \|E_\theta\|_p , \tag{23}$$

where $(e^{i\theta} I - (A + E_\theta))$ is singular. That is, if $r_p(A) \le \varepsilon$ then, there exists a E with $\|E\|_p = \varepsilon$ such that $|\lambda_1(A + E)| = 1$. When A is normal, $r_2(A) = 1 - |\lambda_1(A)|$. In general, when A is nonnormal, $r_2(A) \le 1 - |\lambda_1(A)|$ (Mori, 1990). If $r_2(A) \ll 1 - |\lambda_1(A)|$, the properties of P may be poorly described by the the eigenvalues of A. If the solution of the DALE is known for *any* choice of Q, the results here give lower bounds for the radius of stability, a quantity of considerable interest and study in Robust Stability (Kolla, 1996).

Remark 6. The bounds introduced here can be interpreted in terms of the *pseudospectrum* of A (Trefethen, 1997). The ε-pseudospectrum of A consists of approximate eigenvalues of A and is defined to be the set of complex points $\lambda \in \mathbb{C}$ such that $\|R(\lambda, A)\|_2 \ge \varepsilon^{-1}$. Choosing $\varepsilon = r_2(A)$ gives the largest value of ε such that the ε-pseudospectrum of A lies within the unit circle.

When A is normal, the behavior of the DALE is tied to the spectrum of A. For normal A, P can be large compared to Q only when the spectrum of A is close to the unit circle. The results here show that for general A the behavior of the DALE depends on the pseudospectrum of A and that P can be large only when the pseudospectrum of A is close to the unit circle. The calculation of the pseudospectrum of A consists of calculating $\|R(\lambda, A)\|$ as a function of the complex variable λ, precisely the information needed to compute the bounds presented here. Theoretical and computational techniques developed to calculate and analyze pseudospectra may be used to investigate properties of the solution of the DALE. Efficient methods for computing pseudospectra of large matrices have been developed (Liu, 1997, and references therein).

Remark 7. Denote by B the solution of Eq. (1) for $Q = I$. In Gahinet et al. (1990) the conjecture was made that $\|B\|_2$ is no larger than $r_2^{-2}(A)$. Theorem 3 shows that this conjecture is indeed true. Properties of B can be used to obtain estimates for the solution of the DALE with arbitrary Q. If P is the solution of Eq. (1) then, we have $(P - \gamma B) = A(P - \gamma B) \, A^T + (Q - \gamma I)$. Choosing the extremal values of γ such that $(Q - \gamma I)$ is positive definite or negative definite leads to the matrix bounds

$$\lambda_n(Q) B \le P \le \lambda_1(Q) B. \tag{24}$$

Remark 8. A desirable property of an upper bound is that the upper bound goes to infinity if and only if the solution itself is unbounded. Many upper bounds for the solution of the DALE do not have this property. For example, the matrix upper bound (and eigenvalue

Table 1
Comparison of upper bounds for A given in Eq. (26)

ε	$\|A\|_2$	$\|(1-\sigma_1^2(A))^{-1}A^{\mathrm{T}}A+I\|_2$	$r_2^{-2}(A)$	$(2\pi)^{-1}\int_0^{2\pi}\|R(\mathrm{e}^{i\theta},A)\|^2\,\mathrm{d}\theta$	$\|B\|^2$
10^{-2}	0.992	62.9	15.75	3.77	3.19
10^{-3}	0.9992	625.5	16.0	3.82	3.23
10^{-4}	0.9999	6.25×10^3	16.0	3.82	3.23
10^{-5}	1.0	6.25×10^4	16.0	3.82	3.23

Table 2
Comparison of upper bounds for A given in Eq. (28)

ε	$\|A\|_2$	κ_2	$\kappa_2^2/(1-\|\lambda_1(A)\|^2)$	$r_2^{-2}(A)$	$(2\pi)^{-1}\int_0^{2\pi}\|R(\mathrm{e}^{i\theta},A)\|^2\,\mathrm{d}\theta$	$\|B\|_2$
10^{-2}	5.05	102	1.4×10^4	416.2	77.14	76.11
10^{-3}	5.05	103	1.3×10^6	408.0	76.71	75.68
10^{-4}	5.05	104	1.3×10^8	408.0	76.7	75.67
10^{-5}	5.05	105	1.3×10^{10}	408.0	76.7	75.67

bounds derived from it) (Lee, 1996)

$$P \le \frac{\lambda_n(Q)}{1-\sigma_1^2(A)} A^{\mathrm{T}}A + Q, \quad \sigma_1(A) < 1 \tag{25}$$

becomes unbounded as $\sigma_1(A) \to 1$. However, when A has singular values greater than or equal to one, solutions of the DALE are bounded if the eigenvalues of A lie inside the unit circle. For example, consider

$$A = \begin{bmatrix} 0.5 & 0.75-\varepsilon \\ 0 & 0.5 \end{bmatrix}, \tag{26}$$

and $Q = I$. Table 1 shows that as ε goes to zero, the estimate in Eq. (25) is unbounded while the bounds from Theorem 3 are, like the solution B, well behaved. The lack of tightness in the radius of stability bound is due to replacing an integrand by its maximum value.

Remark 9. Similar misleading behavior can be seen in the upper bound (Gahinet et al., 1990)

$$\|B\| \le \frac{\kappa^2}{1-|\lambda_1(A)|^2}, \tag{27}$$

where κ is the 2-norm condition number of the eigenvectors of A. Note that for A in the previous example $\kappa = \infty$ and Eq. (27) cannot be applied. The conditioning of the eigenvectors of A does not always reflect the behavior of B. Consider the A from Example 3 of Gahinet et al. (1990)

$$A = \begin{bmatrix} 0.5 & 5\varepsilon^2 \\ 5 & 0.5 \end{bmatrix} \tag{28}$$

and $Q = I$. As ε goes to zero, the condition of the eigenvectors of A is unbounded. In this example $\sigma_1(A) > 1$ and the bounds in Eq. (25) cannot be applied. Table 2 shows that though the upper estimate in Eq. (27) becomes unbounded as $\varepsilon \to 0$, the new bounds and the solution B are well-behaved. Again, the overestimate in the $r_2^{-2}(A)$ bound can be traced to replacing the integrand by its maximum value.

Remark 10. The bounds presented here have the pleasant property that the upper bound is infinite if and only if the solution is infinite. This property can be seen from the result (Gahinet et al., 1990)

$$\|B\|_2 \ge \frac{1}{2r_2(A) + r_2^2(A)}. \tag{29}$$

Hence, when $r_2(A)$ goes to zero and the bound in Eq. (13) goes to infinity, then $\|B\|_2$ must be unbounded. This argument can be extended to other matrix norms by noting that all matrix norms are equivalent; if $r(A)$ goes to zero for a particular choice of norm, it goes to zero in all norms (Golub and Van Loan 1996, Section 2.3.2). Likewise, if $\|B\|_2$ is unbounded then, B is unbounded in all matrix norms. Finally, the bounds in Eq. (24) show that if B is unbounded then so is P.

Remark 11. We now present an example where the new bounds are particularly valuable for analyzing a family of problems depending on a parameter. In this example $\sigma_1(A) = 1$ and $\kappa = \infty$ and the previously known bounds in Eqs. (25) and (27) give no information. Consider the dynamics coming from the one-dimensional advection equation

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = 0, \quad 0 \le x \le n \tag{30}$$

with boundary condition $w(0) = 0$. Define $w^d(t)$ to be the n-vector with components $w_i^d(t) = w(x = i, t)$ for

$i = 1, \ldots, n$. The matrix A that advances $w^d(t)$ one time unit is the $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & . & . & 0 \\ 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ .. & .. & . & . & . & . \\ .. & . & . & 1 & 0 & 0 \\ .. & . & . & 0 & 1 & 0 \end{bmatrix}. \tag{31}$$

The eigenvalues of A are all zero, independent of n. A direct calculation of the radius of stability shows that

$$r_{\mathrm{F}}(A) = \left(\frac{n(n+1)}{2}\right)^{-1/2}. \tag{32}$$

The 2-norm radius of stability is more difficult to estimate analytically in closed form. However, there is the lower bound (Ipsen, 1997, Theorem 8.4)

$$r_2(A) \geq \frac{1}{n}. \tag{33}$$

Using Eqs. (13) and (17) gives upper bounds for B (the solution of Eq. (1) with $Q = I$)

$$\operatorname{tr} B \leq \frac{n(n+1)}{2} \quad \text{and} \quad \|B\|_2 \leq n^2. \tag{34}$$

The exact solution is

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ .. & .. & .. & .. & . \\ 0 & . & . & n-1 & 0 \\ 0 & . & . & 0 & n \end{bmatrix} \tag{35}$$

with $\operatorname{tr} B = n(n+1)/2$ and $\|B\|_2 = n$. In this example, $\operatorname{tr} B$ achieves the upper bound. The upper bound for $\|B\|_2$ is an overestimate but captures the behavior that as n goes to infinity, B is unbounded.

4. Conclusions

New upper bounds for solutions of the discrete algebraic Lyapunov are presented. The bounds are applicable with no restrictions on A other than that its eigenvalues lie inside the unit circle. In particular, there are no restrictions on the singular values of A or on the diagonalizability of A. The bounds are given in terms of the resolvent of A, a quantity that is related to the radius of stability of A and the pseudospectrum of A. The upper bounds are computable when the large dimension of A make direct solution of the DALE impossible. The behavior of the new bounds is shown to be more realistic than that of many previously known ones, in the sense that the new bounds go to infinity if and only if the solution of the DALE is unbounded.

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