# Goals, Constraint and Optimality

***David Letson***

*University of Miami*

# Outline

Expected utility framework.

Basic Approach: Kuhn Tucker Conditions

Intermediate products.

Spatial networks

Inventory decisions.

Multiple Objectives.

# Expected Utility

Better than decision trees when many forks exist.

Max subjective expected utility    max CE.

Risk aversion should decline with increasing wealth.

Coefficient of relative risk aversion (CARA):

$$U(W_F) = W_F^{(1-R_r)} / (1 - R_r)$$

# Expected Utility

The shape of the utility function embodies risk preference. Its curvature is expressed as a ratio of its second and first derivatives:

$$R_a = -U''(W) / U'(W)$$

which is the coefficient of absolute risk aversion. The CRRA is obtained by multiplying by wealth:

$$R_r = WR_a$$

CARA does not decline with increasing wealth, as does CRRA. Also, CARA is in units of inverse wealth, which depends on units of currency.

# How to solve a constrained optimization problem.

Three Steps:

- Form the Lagrangean for the problem.

- Differentiate with respect to all variables.

- Form and interpret complementary slackness conditions.

# Step 1: Form the Lagrangean Function

Suppose a problem:

$$\max f(x)$$
$$\text{s.t. } g_i(x) \quad 0,$$
$$x \quad 0,\ i=1,\ldots,n$$

The problem could be one of maximizing farm income subject to the availability of land and other scarce resources. The Lagrangean function for this problem is:

$$L(x,\text{ë}) = f(x) - \text{Ó}\ \text{ë}_i g_i(x)$$

Where ë is a vector of Lagrangean multipliers indicating the marginal values of the resource constraints.

# Steps 2 & 3: Take Partial Derivatives, Form Complementary Slackness Conditions

The problem has a solution if and only if:

(1)    L(x*, ë*)/   x    0 and x*(  L(x*, ë*)/   x) = 0

(2)    L(x*, ë*)/    ë    0 and ë*(  L(x*, ë*)/   ë) = 0

where * indicates the optimal level of the variable. These are also known as the Kuhn-Tucker conditions.

# Example: Commodity Supply & Demand

Suppose we want to maximize total benefits from using some commodity and that

demand price: $P(x_d) = a_{11} + a_{12}x_d$ and

supply price: $Q(x_s) = a_{21} + a_{22}x_s$.

# Problem Formulation

Geometrically speaking, we wish to maximize the area between the demand and supply curves:

$$P(x_d)dx_d \; - \quad Q(x_s)dx_s$$

$$= a_{11}x_d + 0.5a_{12}x_d^2 – a_{21}x_s – 0.5a_{22}x_s^2$$

subject to the constraints:

$x_d – x_s \quad 0$ and $x_d, x_s \quad 0$

# Form the Lagrangean

$L(x_d,x_s,p) = a_{11}x_d + 0.5a_{12}x_d^2 - a_{21}x_s - 0.5a_{22}x_s^2 -$
$p(x_d - x_s)$

The Kuhn-Tucker conditions are:

(1) $p^* \quad a_{11} + a_{12}x_d^*$ and $x_d^*(p^* - a_{11} - a_{12}x_d^*) = 0;$

(2) $p^* \quad a_{21} + a_{22}x_s^*$ and $x_s^*(p^* - a_{21} - a_{22}x_s^*) = 0;$

(3) $x_d^* - x_s^* \quad 0$ and $p^*(x_d^* - x_s^*) = 0;$

(4) $p^*, x_d^*, x_s^* \quad 0.$

# Interpretation of K-T conditions

(1) Demand price does not exceed market price. If a positive amount is demanded, then the demand price will equal the market price.

(2) The supply price does not fall short of the market price. If a positive amount is supplied, then the supply price will equal the market price.

(3) Demand cannot exceed supply. If a positive price is established in the market, then demand will equal supply.

(4) Supply, demand, and price cannot be negative.

# Another Way to Solve Problem?

For all linear programs and many nonlinear ones, there exists a dual problem.

In the example, we formed a maximand by integrating over demand price and supply price functions, with the constraint that demand cannot exceed supply.

Instead, we could have formed a minimand by integrating over the direct demand and supply functions, subject to a zero profit constraint.

# Rummaging in the Toolbox

Management science has many modeling approaches
that allow study of integrated production systems.

Among these are:

- Transportation problem: spatial networks.
- Activity analysis: vertical integration of production stages.
- Warehouse problem: time path of inventory.
- Goal programming: multiple objectives.

A few examples will illustrate how these tools can be combined to study the integrated decision making underlying climate forecast use.

# The Vertical Dimension

Production often sequenced into chains that also make use of intermediate products and spatial distribution networks.

Many animal products and specialty crops are produced this way.

Honduran pineapple example from Thore (1991).

Three harvest regions of varying productivity.

Two canning plants, one of which is newer and more capital intensive.

# Honduran Pineapple Example

Supply price functions:

$Q_1 = 5w_1 - 20000$
$Q_2 = w_2 - 5000$
$Q_3 = 2w_3 - 12000$

Input Costs:

labor: $1.25/hour
capital: $0.40/hour

Input requirements:

| | labor | capital | pineapple |
|---|---|---|---|
| old plant | 0.08 | 1.2 | 16 |
| new plant | 0.04 | 2.0 | 16 |

Transportation Costs($/lb):

| From Region | To New Plant | To Old Plant |
|---|---|---|
| 1 | 0.02 | 0.04 |
| 2 | 0.015 | 0.03 |
| 3 | 0.04 | 0.01 |

World price of pineapples is $12.50/case.

Solve for supply, distribution and pricing of harvested pineapples.

# Forming the Problem Statement

Denoting harvest as w, canning as z, and shipments as t, the problem is:

Max $12.5(z_{new} + z_{old}) +$ (revenue)

$20000w_1 - 2.5w_1^2 + 5000w_2 – 0.5w_2^2 + 12000w_3 – w_3^2$ (harvest costs)

$– 1.25(0.4z_{new} + 0.08z_{old})$ (labor costs)

$– 0.40(2.0z_{new} + 1.2z_{old})$ (capital costs)

$– (0.2t_{1new}+0.015t_{2new}+0.4t_{3new}+0.04t_{1old}+0.03t_{2old}+ 0.01t_{3old})$ (shipping costs)

# Subject to:

$16z_{new} - t_{1new} - t_{2new} - t_{3new} \quad 0$
$16z_{old} - t_{1old} - t_{2old} - t_{3old} \quad 0$
(canning requirements)

$t_{1new} + t_{1old} - w_1 \quad 0$ (harvest balances)
$t_{2new} + t_{2old} - w_2 \quad 0$
$t_{3new} + t_{3old} - w_3 \quad 0$

$z, t, w \quad 0$ (non-negativity)

# Results:

Supplies in the three harvesting regions are: 4000, 5000, 6000 lbs/day.

Harvest in region 1 shipped to new plant.
Harvest in regions 2 & 3 shipped to old plant.

New plant exports 250 cases/day.
Old plant exports 687.6 cases/day.

Price that growers receive at new plant: $0.728/lb.
Price that growers receive at old plant: $0.745/lb.

# The Temporal Dimension

Many production decisions involve more than the current time period.

Intertemporal decisions include plans for grain storage and animal stocking.

Inventory planning example from Thore (1991).

# Grain Storage Problem

Operator plans purchases (w) and sales (x) over next five months.

Initial inventory and desired ending inventory both are 300 units.

Supply price and demand price functions expected in this time:

| Month | Supply Price | Demand Price |
|---|---|---|
| 1 | $-30 + 2w_1$ | $40 – 0.75x_1$ |
| 2 | $-25 + 2w_2$ | $42 – 0.75x_2$ |
| 3 | $-20 + 2w_3$ | $44 – 0.75x_3$ |
| 4 | $-15 + 2w_4$ | $46 – 0.75x_4$ |
| 5 | $-10 + 2w_5$ | $48 – 0.75x_5$ |

Warehouse capacity = 310 units.
Unit storage costs = \$0.8/unit/month.
Commodity does not deteriorate in storage.

# Problem Statement

Maximize

$40x_1 + 42x_2 + 44x_3 + 46x_4 + 48x_5$
$- 0.375(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)$
(integrated demand price functions)

$- (30w_1 - 25w_2 - 20w_3 - 15w_4 - 10w_5)$
$- (w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2)$
(integrated supply price functions)

$- 0.8(I_1 + I_2 + I_3 + I_4)$
(inventory costs)

# Subject to Constraints:

| | | |
|---|---|---|
| $x_1 - w_1 + I_1$ | 300 | (inventory |
| $x_2 - w_2 - I_1 + I_2$ | 0 | accumulation) |
| $x_3 - w_3 - I_2 + I_3$ | 0 | |
| $x_4 - w_4 - I_3 + I_4$ | 0 | |
| $x_5 - w_5 - I_4$ | -300 | |
| | | |
| $I_1 , I_2 , I_3 , I_4$ | 310 | (warehouse |
| | | capacity) |
| x, w, I | 0 | (non-negativity) |

# Results

| Month | Purchases | Sales | Inventory | Prices |
|---|---|---|---|---|
| 1 | 27.32 | 20.47 | 306.85 | 24.65 |
| 2 | 25.22 | 22.07 | 310 | 25.45 |
| 3 | 23.27 | 23.27 | 310 | 26.55 |
| 4 | 21.32 | 24.47 | 306.85 | 27.65 |
| 5 | 19.22 | 26.07 | 300 | 28.45 |

Notes:

Price rises each month by the amount of the storage cost ($0.8).

In months 2 & 3 there is also a charge for meeting the capacity limit of the warehouse ($0.3).

# Multiple Objectives

Sometimes problem has multiple and perhaps competing objectives.

Goal programming examines tradeoffs between objectives with the use of “soft” constraints.

Feedlot & whaling examples from Thore (1991).

# (1) Cattle Feedlot Example

Production occurs at four feedlots in a region, with supply price functions ($/000 tons):

$P_a(w_a) = -10 + 1.8w_a$ $P_c(w_c) = -4 + 1.6w_c$
$P_b(w_b) = -8 + 2w_b$ $P_d(w_d) = -8 + 1.8w_d$

Cattle shipped to two packers having demands 40 and 48. Rail costs are $0.002/mile/000 tons. Distances are:

| | To Packer 1 | To Packer 2 |
|---|---|---|
| Feedlot A | 600 | 1000 |
| Feedlot B | 1000 | 800 |
| Feedlot C | 200 | 1500 |
| Feedlot D | 500 | 750 |

Environmental restriction: A = 11.5, B = 20, C = 40 and D = 16.

# Consider Two Policy Alternatives

*Environmental Penalties*

| Region | Policy i | Policy ii |
|---|---|---|
| A | 444 | 222 |
| B | 222 | 444 |
| C | 888 | 888 |
| D | 666 | 666 |

# Problem Statement

Minimize:

$-10w_a + 0.9w_a^2 – 8w_b + w_b^2 – 4w_c + 0.8w_c^2 + -9w_d + 0.9w_d^2$

(production costs)

$+ 1.2t_{a1} + 2t_{b1} + 0.4t_{c1} + t_{d1} + 2t_{a2} + 1.6t_{b2} + 3t_{c2} + 1.5t_{d2}$

(shipping costs)

$+ 444e_1 + 222e_2 + 888e_3 + 666e_4$

(environmental penalties)

# Subject to Constraints:

$t_{a1} + t_{b1} + t_{c1} + t_{d1} \quad 40$ (demand balances)
$t_{a2} + t_{b2} + t_{c2} + t_{d2} \quad 48$

$w_a - t_{a1} - t_{a2} \quad 0$ (supply balances)
$w_b - t_{b1} - t_{b2} \quad 0$
$w_c - t_{c1} - t_{c2} \quad 0$
$w_d - t_{d1} - t_{d2} \quad 0$

$w_a - e_a \quad 11.5$ (goals)
$w_b - e_b \quad 20$
$w_c - e_c \quad 40$
$w_d - e_d \quad 16$

$w, t, e \quad 0$ (non-negativity)

# Results for Policy i

Optimal production and shipments:

$w_a^* = 11.5$, $w_b^* = 20.5$, $w_c^* = 40$, $w_d^* = 16$

$t_{a2}^* = 11.5$, $t_{b2}^* = 20.5$, $t_{c1}^* = 40$, $t_{d2}^* = 16$

Demand prices at packers: $p_1^* = 255.8$, $p_2^* = 256.6$

Market prices at feedlots:

$p_a^* = 254.6$, $p_b^* = 255.0$, $p_c^* = 255.4$, $p_d^* = 255.1$

Duals on goals provide optimal tax:

$s_a^* = 243.9$, $s_b^* = 222.0$, $s_c^* = 195.4$, $s_d^* = 234.3$

In each region, supply price plus tax = market price.

# Results for Policy ii

Optimal production and shipments:

$w_a^* = 12$, $w_b^* = 20$, $w_c^* = 40$, $w_d^* = 16$
$t_{a2}^* = 12$, $t_{b2}^* = 20$, $t_{c1}^* = 40$, $t_{d2}^* = 16$

Demand prices at packers: $p_1^* = 234.8$, $p_2^* = 235.6$

Market prices at feedlots:

$p_a^* = 233.6$, $p_b^* = 234.0$, $p_c^* = 234.4$, $p_d^* = 234.1$

Duals on goals provide optimal tax:

$s_a^* = 222.0$, $s_b^* = 202.0$, $s_c^* = 174.4$, $s_d^* = 213.3$

In each region, supply price plus tax = market price.

# (2) Whaling Example

Whaling occurs in two Antarctic regions. The distances from hunting regions to ports are:

| | From Port A | From Port B |
|---|---|---|
| From region 1 | 2000 | 1750 |
| From region 2 | 3000 | 2500 |

Transportation costs are $0.001 per mile and thousand lbs of processed whale meat.

Government goal for limiting catch per region is 43750 lbs.

There are two vessel types, whalers and factory ships. Their input requirements are:

| | Whalers | Factory Ships |
|---|---|---|
| Capital | 0.8 | 1.2 |
| Fuel | 1.0 | 0.5 |
| Labor | 1.0 | 0.8 |

# (2) Whaling Example

Region 1 is closer to both ports but has been more heavily fished in the past and has greater input costs per thousand lbs of whale meat:

| | Region 1 | Region 2 |
|---|---|---|
| Capital | $6 | $5 |
| Fuel | 10 | 9 |
| Labor | 5 | 4 |

Total demand for whale meat is 300,00 lbs in port A and 60,000 in port B.

Two options:

(a) region 1 is priority for protection, with penalties set to $999 and $666 per thousand lbs caught above target;

(b) region 2 is priority with the penalties equal to $666 and $999.

Find optimal catch, distribution and prices.

# Problem Statement

$\min 6(0.8z_{11} + 1.2z_{12}) + 5(0.8z_{21} + 1.2z_{22})$

(capital costs)

$+ 10(z_{11} + 0.5z_{12}) + 9(z_{21} + 0.5z_{22})$

(fuel costs)

$+ 5(z_{11} + 0.8z_{12}) + 4(z_{21} + 0.8z_{22})$

(labor costs)

$+ 2t_{1a} + 1.75t_{1b} + 3t_{2a} + 2.5t_{2b}$

(transportation costs)

$+ 999e_1 + 666e_2$

(penalties)

# Subject to Constraints:

$t_{1a} + t_{2a} \quad 30$ (demand at ports)

$t_{1b} + t_{2b} \quad 60$

$z_{11} + z_{11} - t_{1a} - t_{1b} \quad 0$ (shipments from each

$z_{21} + z_{22} - t_{2a} - t_{2b} \quad 0$ hunting region)

$z_{11} + z_{12} + d_1 - e_1 = 0$ (goals to limit harvest)

$z_{21} + z_{22} + d_2 - e_2 = 0$

$z, t, e \quad 0$ (non-negativity)

# Results

Optimal catch and distribution:

Region 1 catch is 43750, 30000 of which is shipped to port A and 13750 to port B.

Region 2 catch is 46250, all of which is sent to port B. Note harvest goal exceeded by 2500.

Notes:

Only factory ships would be used in both regions.

Effect of switching protection priorities changes the location of harvest but not overall amount.

# More Results

Prices: Price is sum of input costs and penalty/tax.

| | Region 2 | Region 1 |
|---|---|---|
| Capital | $5 X 1.2 = 6 | $6 X 0.8 = 4.8 |
| Fuel | 9 X 0.5 = 4.5 | 10 X 1.0 = 10 |
| Labor | 4 X 0.8 = 3.2 | 5 X 1.0 = 5 |
| Pigovian tax | 666 | 660 |
| Transportation to port B | 2.5 | 1.75 |
| Total | $682 | $682 |

Producers in region 1 could avoid the penalty but would have to pay a tax nearly as large, if goals are to be met.

Effects of harvest penalty: higher consumer prices, likely lower harvest (not shown here).

# Conclusions

*Ex ante* evaluation of climate forecast use requires that we model a range of decisions.

Constrained optimization useful for considering:

- Risk aversion
- Intermediate products
- Distribution networks
- Inventory decisions
- Multiple objectives.

Knowing how your nonlinear problem is solved will help get more accurate answers from your software.